§1 Interpolation

§1.3 Interpolation Error Estimates

MA378/531 – Numerical Analysis II ("NA2")

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Augustin-Louis Cauchy (1789–1857), Paris, France. He was a pioneer of analysis, in particular in introducing rigour into calculus proofs. He founded the fields of complex analysis and the study of permutation groups.

In our last example, we wrote down the polynomial of degree $n = 2$ interpolating $f(x) = e^x$ at $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$.

We now want to investigate how, in general, error in polynomial interpolation depends on

(i) the function (and its derivatives)

(ii) the number of points used (or, equivalently, degree of the polynomial used).

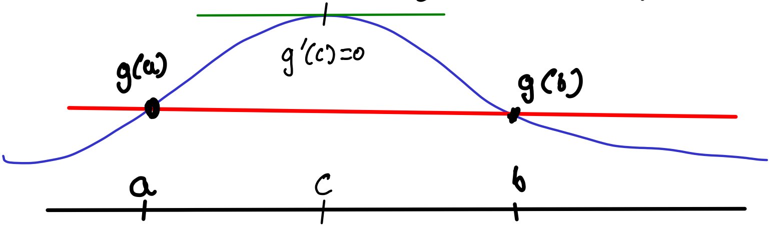
The main ingredient we need to the following theorem.

Theorem (Rolle's Theorem)

Let g be a function that is continuous and differentiable on the interval $[a, b]$. If $g(a) = g(b)$, then there is at least one point c in (a, b) where $g'(c) = 0$.

Tangent to g has slope 0 at c.

Our "proof" is by picture:[1]



[1] One can easily deduce Rolle's Theorem from the Mean Value Theorem (MVT). But since the standard proof of the MVT uses Rolle's Theorem, that would be cheating

Error estimate for n=0

The simplest case is when $n = 0$, so the interpolant is a constant, i.e., it is p_0 interpolating a function f at a point x_0. Here is one way we can deduce the *interpolation error*.

Since p_0 is a constant, and $p_0(x_0) = f(x_0)$ we have $p_0(x) = f(x_0)$ for all x.

Take any $x \neq x_0$, define the function

$$g(t) = f(t) - p_0(t) - \left[\frac{f(x) - p_0(x)}{x - x_0}\right](t - x_0).$$

Then

$$g(x_0) = \underbrace{f(x_0) - p_0(x_0)}_{0} - \left[\frac{f(x) - p(x)}{x - x_0}\right]\underbrace{(x_0 - x_0)}_{0} = 0$$

Also

$$g(x) = f(x) - p_0(x) - \frac{f(x) - p_0(x)}{\cancel{x - x_0}}\cancel{(x - x_0)} = 0$$

So g is zero at two points: $t = x_0$ & $t = x$.

It is important to understand what this formula is telling us:

First, we don't know τ, but it is in $[x_0, x]$.

If f is constant, the error is zero (as it should be!) because $f'(x)=0 \ \forall x$.

The larger f' is, the larger the error.

The larger $|x - x_0|$, the larger the error.

Finally, although we assumed $x \neq x_0$, the formula holds in this case.

We will use this Rolle's Theorem to prove the most important theorem of NA2; it is used repeatedly through-out the course. It's often called the *Polynomial Interpolation Error Theorem*, but we'll call it *Cauchy's Theorem* for short.

First, we need to define an important polynomial.

Definition (Nodal Polynomial)

The **Nodal Polynomial** π_{n+1} associated with the interpolation points that $a = x_0 < x_1 < \cdots < x_n = b$ is

$$\pi_{n+1}(x) = (x - x_0)(x - x_1) \dots (x - x_n) = \prod_{i=0}^{n} (x - x_i).$$

Theorem (Cauchy, 1840)

Suppose that $n \geq 0$ and f is a real-valued function that is continuous and defined on $[a, b]$, such that the derivative of f of order $n+1$ exists and is continuous on $[a, b]$. The p_n be the polynomial of degree n that interpolates f at the $n+1$ points $a = x_0 < x_1 < \cdots < x_n = b$. Then, for any $x \in [a, b]$ there is a $\tau \in (a, b)$ such that

$f^{(k)} = \frac{d^k f}{dx^k}$ $\qquad f^{(0)} = f$

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\pi_{n+1}(x). \tag{1}$$

Proof. First, suppose that $x = x_i$ for some $i = 0, 1, \dots, n$.

Then $f(x_i) = p_n(x_i)$ so $f(x_i) - p_n(x_i) = 0$.

Also $\pi_{n+1}(x_i) = 0$. So the formula holds.

Next, take any $x \neq x_0$ and define the auxillory function g

$$g(t) = f(t) - p_n(t) - \left[\frac{f(x) - p_n(x)}{\pi_{n+1}(x)}\right] \pi_{n+1}(t).$$

Then $g(x_i) = \underbrace{f(x_i) - p_n(x_i)}_{=0} - \left[\frac{f(x) - p_n(x)}{\pi_{n+1}(x)}\right] \underbrace{\pi_{n+1}(x_i)}_{=0}$

$= 0$. So g has $n+1$ zeros.

Also, $g(x) = f(x) - p_n(x) - \left[f(x) - p_n(x)\right] \dfrac{\pi_{n+1}(x)}{\pi_{n+1}(x)} = 0$,

So, in fact, g has $n+2$ zeros:

$\{x_0, x_1, \ldots, x_n, x\}$.

Now by Rolles Theorem, between every pair of adjacent zeros, g' has a zero. There are $n+1$ of these.

By repeated application of Rolle's Theorm,

g'' has n distint zeros,

g''' has $n-1$ " ",

$\vdots$

$g^{(n+1)}$ has at least one zero. We'll call this point τ.

That is, there is a point $\tau \in [a, b]$ such that $g^{(n+1)}(\tau) = 0$. Thus,

$$f^{(n+1)}(\tau) - p_n^{(n+1)}(\tau) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}^{(n+1)}(\tau) = 0.$$

But p_n is a poly of degree n, so $p_n^{(n+1)}(x) = 0$ for all x. Furthermore

$\pi_{n+1}(x) = (x - x_0)(x - x_1) \cdots (x - x_{n+1})$. This is can be written as $\pi_{n+1}(x) = x^{n+1} + \sum_{i=0}^{n} c_i x^i$

So $\pi_{n+1}^{(n+1)}(x) = (n+1)!$ So now we have

$$f^{(n+1)}(\tau) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} (n+1)! = 0.$$

Rearrange to finish.

Example

In an earlier example, we wrote down the Lagrange form of the polynomial, p_2, that interpolates $f(x) = e^x$ at the points $\{-1, 0, 1\}$. Give a formula for $e^x - p_2(x)$.

Here $n=2$, so $f(x) - p_2(x) = \dfrac{f'''(\tau)}{3!}\pi_3(x)$

$$= \frac{e^\tau}{6}(x-(-1))(x-0)(x+1)$$

$$= (e^\tau)\tfrac{1}{6}(x^3 - x),$$

Also,

$$|f(x) - p_2(x)| \leq \tfrac{1}{6}e(x^3 - x)$$

since $e^\tau \leq e^1$ for all $\tau \in [-1, 1]$

Usually (and as in the above example), we can't calculate $f(x) - p_n(x)$ exactly from Formula (1), because we have no way of finding τ. However, we are typically not so interested in what the error is at some given point, but what is the maximum error over the whole interval $[x_0, x_n]$. That is given by:

Corollary

Define

$$M_{n+1} = \max_{\cancel{\sigma \in [x_0, x_n]}} |f^{(n+1)}(\sigma)|.$$

$x_0 \leq \sigma \leq x_n.$

Then

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)|. \tag{2}$$

Example

Let p_1 be the polynomial of degree 1 that interpolates a function f at distinct points x_0 and x_1. Letting $h = x_1 - x_0$, show that

$$\max_{x_0 \leq x \leq x_1} |f(x) - p_1(x)| \leq \frac{1}{8} h^2 M_2.$$

To see this is true, recall that, for all $x \in [x_0, x_1]$,

$$|f(x) - p_1(x)| \leq \frac{1}{2} \underbrace{\max_{x_0 \leq \sigma \leq x_1} |f''(x)|}_{M_2} \; \underbrace{|(x - x_0)(x - x_1)|}_{\pi_2(x)}.$$

Note that

$$|\pi_2(x)| = |x^2 - x(x_0 + x_1) + x_0 x_1|$$

It takes its max where $\pi_2'(x) = 0$, ie, $2x - (x_0 + x_1) = 0$.

ie at $x^* = \dfrac{x_0 + x_1}{2}$. Then $\pi_2(x^*) = \left(\dfrac{x_0 + x_1}{2} - x_0\right)\left(\dfrac{x_0 + x_1}{2} - x_1\right)$