§1 Interpolation

§1.4 Computing the Polynomial Interpolant

MA378/531 – Numerical Analysis II ("NA2")

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Stamps commemorating the scientific achievements of Isaac Newton from (left to right) Germany, Poland and Vietnam. Source: `http://jeff560.tripod.com/stamps.html`

Sir Issac Newton (1642–1726), Great Britain. One of the greatest scientists this world has ever produced.

The Lagrange form of the polynomial interpolant has great theoretical importance. It is also useful for $n = 0, 1, 2$, say.

It practice, for larger n, it is easy to write down the Lagrange form of the polynomial interpolant, but it is tedious (and computationally expensive) to work with. Evaluating each of the

$$L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

at a given value of x requires $2n - 1$ multiplications. So therefore computing

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x) = \sum_{i=0}^{n} \left(y_i \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \right),$$

for a given value of x requires $2n^2 + 2n$ multiplications.

On the other hand, suppose we have p_n in the standard form

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

$= \sum_{i=0}^{n} i.$

At first glance it seems that this takes $(n^2 - n)/2$ multiplications, which is only a little better. However there is a faster way of doing this called *synthetic division*. As an example we'll consider how this works for the case $n = 3$. For the general case, please have a look at Lecture 19 of Stewart's "After notes".

In summary,

- The Lagrange form is easy to find and difficult to evaluate.
- The standard form (i.e., $p_n = a_0 + a_1 x + \cdots + a_n x^N$), is hard to find, but easy to evaluate.
- So we want to find a form for the polynomial is easy to construct *and* efficient to evaluate.

There are plenty of possibilities – too many to mention. If you are interested, read the next section on the *Newton Form*.

This section is just for your information: it will not be covered in class or be part of any assessment.

(These notes are based on §2.1.3 of Stoer and Bulirsch and on Lecture 19 of Stewart's "Afternotes...").

Suppose we were to write the interpolating polynomial p_n as

$$\begin{aligned} p_n(x) = a_0 + a_1(x - x_0) \\ + a_2(x - x_0)(x - x_1) \\ + a_3(x - x_0)(x - x_1)(x - x_2) \\ + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}). \end{aligned}$$

Then it could actually be evaluated as

$$\begin{aligned} p_n(x) = a_0 + (x - x_0)\Big(a_1 + (x - x_1)\big(a_2 + (x - x_2)(a_3 + \\ (\cdots + a_n(x - x_{n-1}))\cdots\big)\Big) \end{aligned} \quad (1)$$

This is the **Newton Form of the Interpolating Polynomial**.

We can simplify the representation of

$$\begin{aligned} p_n(x) = a_0 + a_1(x - x_0) \\ + a_2(x - x_0)(x - x_1) \\ + a_3(x - x_0)(x - x_1)(x - x_2) \\ + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}). \end{aligned}$$

Let π_k be the nodal polynomial $\pi_k = \prod_{i=0}^{k}(x - x_i)$. Then the Newton form is

$$p_n(x) = a_0 + a_1\pi_0 + a_2\pi_1 + a_3\pi_2 + \cdots + a_n\pi_{n-1}.$$

So how do we find the coefficients? Let

$$F[x_i] = f(x_i),$$

$$F[x_i, x_{i+1}] = \frac{F[x_{i+1}] - F[x_i]}{x_{i+k} - x_i}$$

$$\vdots$$

$$F[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{F[x_{i+1}, x_{i+2}, \dots x_{i+k}] - F[x_i, x_{i+1}, \dots x_{i+k-1}]}{x_{i+1} - x_i}$$

Then

$$a_0 = f[x_0], \qquad a_1 = F[x_0, x_1], \quad \dots$$

$$a_n = F[x_0, x_1, \dots x_n].$$

Or, if you prefer:

$$p_n(x) = F[x_0] + \sum_{k=1}^{n} F[x_0, x_1, \dots, x_k]\pi_k(x). \qquad (2)$$

It is not hard to show that the formula

$$p_n(x) = F[x_0] + \sum_{k=1}^{n} F[x_0, x_1, \ldots, x_k]\pi_k(x).$$

yields the interpolating polynomial. The argument is inductive. Before we do that, we will demonstrate that it works for a particular example.

Example

Write down the Newton form of the polynomial p_2 that interpolates e^x at $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$.

Solution: (For the Newton form)

	$F[x_i]$	$F[x_i x_{i+1}]$	$F[x_i x_{i+1} x_{i+2}]$
x_0	1		
		$\mathrm{e} - 1$	
x_1	e		$\frac{1}{2}(\mathrm{e}^2 - 2e + 1)$
		$\mathrm{e}^2 - \mathrm{e}$	
x_2	e^2		

This gives

$$p_2(x) = F[x_0] + (x - x_0)\big(F[x_0x_1] + (x - x_1)F[x_0x_1x_2]\big),$$
$$= 1 + x\big((e-1) + (x-2)\frac{1}{2}(e^2 - 2e + 1)\big).$$

Written in the form of (2) above, this is

$$p_2(x) = 1 + (e-1)(x) + \frac{1}{2}(e^2 - 2e + 1)(x)(x-1).$$

One can quickly check that this

1. is a polynomial of degree n,
2. interpolates e^x at $x = 0$, $x = 1$ and $x = 2$.

It is not hard to show that the formula

$$p_n(x) = F[x_0] + \sum_{k=1}^{n} F[x_0, x_1, \dots, x_k]\pi_k(x).$$

yields the interpolating polynomial. The argument is inductive, and given in detail in Section 1.4 of the online notes

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