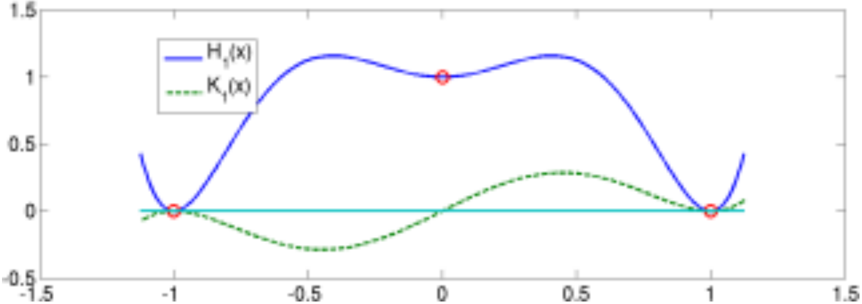
§1 Interpolation

§1.5 Hermite Interpolation

MA378/531 – Numerical Analysis II ("NA2")

January 2017



Hermite interpolation is a variant on the standard Polynomial Interpolation Problem: we seek a polynomial that not only agrees with a given function f at the interpolation points, but its first derivative also matches f' at those points. We are not that interested in this problem for its own sake, but the idea recurs again in the sections in piecewise polynomial interpolation and Gaussian quadrature.

Charles Hermite, France, 1822–1901. A part from this form of interpolation, his contributions to mathematics included the first proof that e is transcendental. His methods were later used to show that π is transcendental.

Formally, the problem is

The Hermite Polynomial Interpolation Problem (HPIP) *Given a set of interpolation points $x_0 < x_1 < \cdots < x_n$ and a continuous, differentiable function f, find $p_{2n+1} \in \mathcal{P}_{2n+1}$ such that*

$$p_{2n+1}(x_i) = f(x_i) \quad \text{and} \quad p'_{2n+1}(x_i) = f'(x_i).$$

for $i = 0, 1, \ldots, n.$

It is not hard to see that, if there is a solution to this problem, then it is unique: Suppose that there are two solutions: p & q. Set $r(x) = p(x) - q(x)$, so $r \in \mathbb{P}_{2n+1}$ & $r(x_i) = f(x_i) - f(x_i) = 0$. So r has $n+1$ zeros: $x_0, x_1, \ldots, x_n$.

See board.

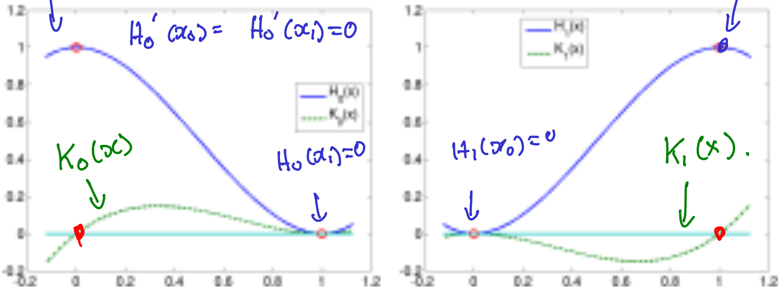
It is possible to solve this problem using an extension of the Lagrange Polynomial approach. Given the usual Lagrange Polynomials, $\{L_i\}$, for $i = 0, \ldots, n$, let

$$H_i(x) = [L_i(x)]^2(1 - 2L_i'(x_i)(x - x_i)),$$

$H_0(x_0)=1$

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

$H_1(x_1)=1$.



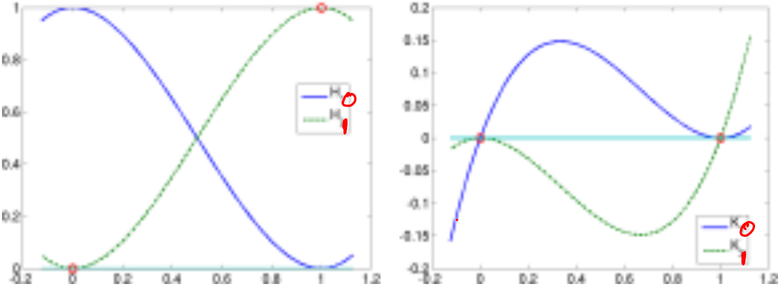
Hermite bases functions H_0, K_0 (left) and H_1, K_1 (right) for $n = 1$, $x_0 = 0$ and $x_1 = 1$

Constructing Hermite Interpolants

It is possible to solve this problem using an extension of the Lagrange Polynomial approach. Given the usual Lagrange Polynomials, $\{L_i\}$, for $i = 0, \ldots, n$, let

$$H_i(x) = [L_i(x)]^2(1 - 2L_i'(x_i)(x - x_i)),$$

$$K_i(x) = [L_i(x)]^2(x - x_i).$$



Hermite bases functions H_0, H_1 (left) and K_0, K_1 (right) for $n = 1$, $x_0 = 0$ and $x_1 = 1$

The Hermite basis functions

$$H_i(x) = [L_i(x)]^2(1 - 2L_i'(x_i)(x - x_i)),$$
$$K_i(x) = [L_i(x)]^2(x - x_i).$$

We can show that, for $i, k = 0, 1, \ldots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad H_i'(x_k) = 0 \ \forall k$$

To see this, recall that $L_i(x_k) = \begin{cases} 1 & i=k \\ 0 & i \neq k. \end{cases}$

So, $H_i(x_i) = [\underbrace{L_i(x_i)}_{1}]^2 (1 - 2L_i'(x_i)(\underbrace{x_i - x_i}_{0})) = 1.$

If $i \neq k$

$H_i(x_k) = [\underbrace{L_i(x_k)}_{0}]^2 (1 - 2L_i'(x_i)(x_k - x_i)] = 0.$

The Hermite basis functions

$$H_i(x) = [L_i(x)]^2(1 - 2L_i'(x_i)(x - x_i)),$$
$$K_i(x) = [L_i(x)]^2(x - x_i).$$

And that, for $i, k = 0, 1, \dots n$,

$$K_i(x_k) = 0, \qquad K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

To show this, see Exer 1.16.

Armed with these identities, we can now show that the solution to the HPIP exists and is given by [finished here 25/1/17]

$$p_{2n+1}(x) = \sum_{i=0}^{n} \left(f(x_i) H_i(x) + f'(x_i) K_i(x) \right).$$

Also a homework exer. But to get started...

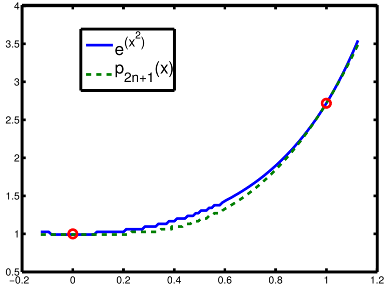
First note that each $L_i(x)$ is a polynomial of degree n. So $[L_i(x)]^2$ is a poly of degree $2n$. It follows that p_{2n+1} is a poly of degree $2n+1$. Next

$$p_{2n+1}(x_j) = \sum_{i=0}^{n} \left(f(x_i) H_i(x_j) + f'(x_i) K_i(x_j) \right)$$

$$= f(x_j) \quad \text{because} \ldots$$

Example

Find the polynomial of degree 3 that interpolates $\exp(x^2)$, and its first derivative, at $x_0 = 0$ and $x_1 = 1$. (See below).



From the formula on the prev slide:

$$p_3(x) = f(x_0)H_0(x) + f(x_1)H_1(x) + f'(x_0)K_0(x) + f'(x_1)K_1(x).$$

$L_0(x) = -(x-1)$ & $L_1(x) = x$.

Then $H_0(x) = [L_0(x)]^2(1 - 2L_0'(x_0)(x-x_0)) = (x-1)^2(1+2x)$.

$H_1(x) = [L_1(x)]^2(1 - 2L_1'(x_1)(x-x_1)) = x^2(1-2(x-1))$

$K_0(x) = [L_0(x)]^2(x-x_0) = (x-1)^2 x$ $\quad K_1(x) = x^2(x-1)$.

... see board.

Theorem

Let be a real-valued function that is continuous and defined on $[a, b]$, such that the derivatives of f of order $2n + 2$ exist and are continuous on $[a, b]$. Let p_{2n+1} be the Hermite interpolant to f. Then, for any $x \in [a, b]$ there is an $\tau \in (a, b)$ such that

$$f(x) - p_{2n+1}(x) = \frac{f^{(2n+2)}(\tau)}{(2n+2)!}[\pi_{n+1}(x)]^2.$$

We won't do a proof of this in class. However, later in this course we'll be interested in the particular example of finding p_3 the cubic Hermite Polynomial Interpolant to a function f at the points x_0 and x_1. Also, see exercises...

Finished.

30/1/17