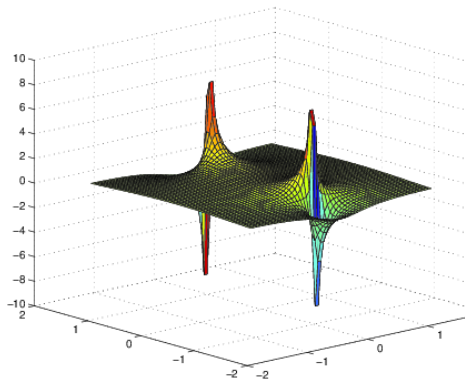


§1 Interpolation
**§Wrap-up:
Convergence &
Runge's Example**

MA378/531 – Numerical
Analysis II (“NA2”)

January 2017



These slides are licensed under [CC BY-SA 4.0](#)

The celebrated Weierstrass approximation theorem states that, given f and a positive number ε , there is a polynomial p such that

$$\max_{x \in [a, b]} |f(x) - p(x)| := \|f - p\|_{\infty} \leq \varepsilon.$$

Now suppose that f is a continuous function on $[a, b]$ and that $\{p_n\}_{n=0}^{\infty}$ is a sequence of polynomials that interpolate f at $n + 1$ equally spaced points. One might be inclined to believe that

$$\lim_{n \rightarrow \infty} \|f - p_n\|_{\infty} = 0.$$

Another way of thinking about this is recalling the error bound:

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)|$$

we might expect that

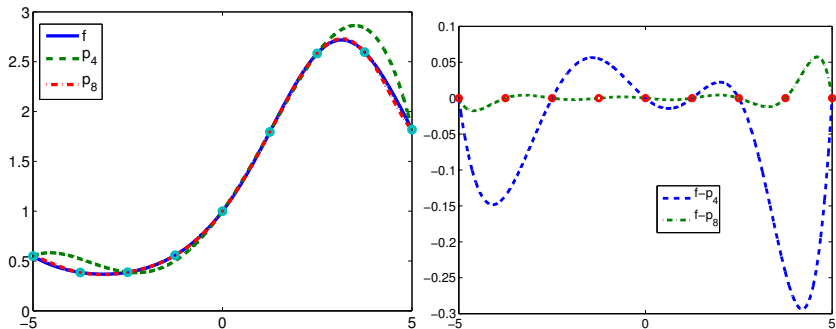
$$\lim_{n \rightarrow \infty} \max_{x \in [a,b]} \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)| = 0.$$

In other words, we might think that, in order to find an interpolating polynomial that is as accurate as we would like, we just need to choose large enough n .

And **some times** this is true. For example, suppose that $a = -5$, $b = 5$, and $f(x) = e^{\sin(x/2)}$. In Table 1 the errors for successive interpolants are shown.

Table: Errors in polynomial interpolants to $e^{\sin(x/2)}$ on $[-5, 5]$

n	$\ f - p_n\ _\infty$
2	1.27e-00
4	2.94e-01
6	8.39e-02
8	5.75e-02
16	1.07e-03



*Polynomial interpolants to $e^{\sin(x/2)}$ on $[-5, 5]$, and their errors
(right)*

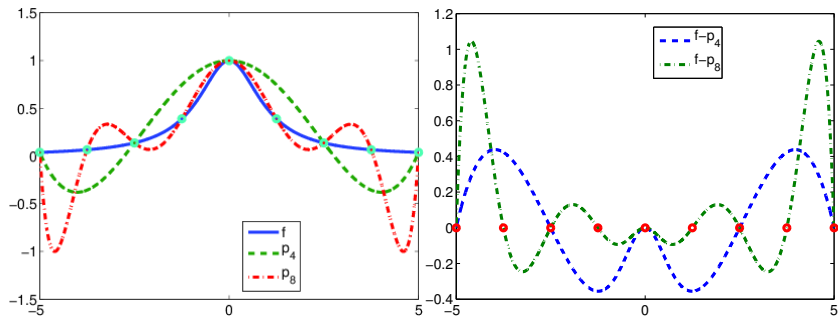
However, there is a famous example of a simple function that cannot be successfully interpolated in this manner

Runge's Example

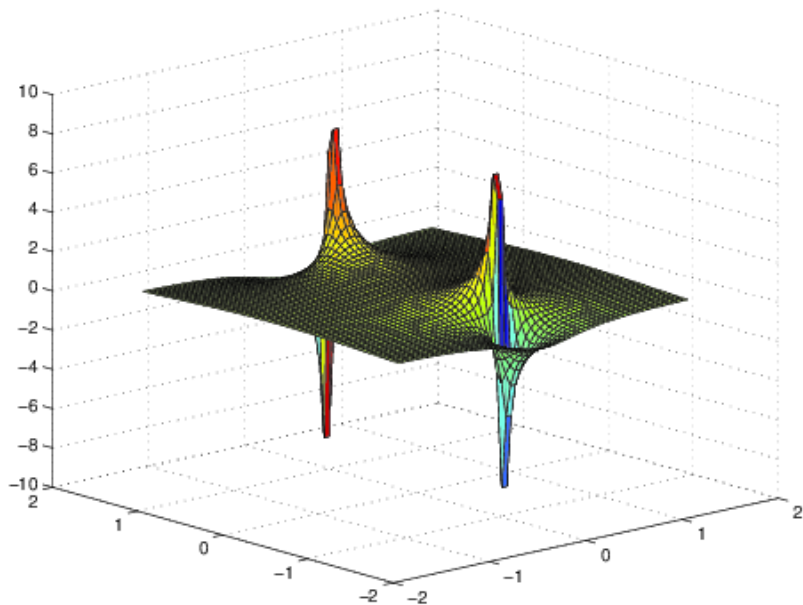
$$f(x) = \frac{1}{1+x^2} \quad \text{on } [-5, 5].$$

Errors for some n are shown below. Notice they *increase* with n .

n	$\ f - p_n\ $
2	0.65
4	0.44
6	0.62
8	1.05
16	14.39
20	59.66
22	122.91
24	257.21



Polynomial interpolants to $1/(1+x^2)$ on $[-5, 5]$



So now it looks like polynomial interpolation is bad, at least on equidistant points. However, our experience in Lab 1 might lead us to be more optimistic: we were able to find a set of points that made the approximation as good we wanted (until round-off error dominated).

Unfortunately, just because we have a good set of points for interpolating one particular function, it does not follow that that set is good for every continuous function: this is ***Faber's Theorem***. This has often led numerical analysts to abandon the idea of interpolation by high-order polynomials completely.

However, there is a set of points that are useful, if f is smooth enough: the **Chebyshev** points of Lab 1. If you are interested, there read the essay ***Inverse Yogiisms*** by Lloyd N. (Nick) Trefethen, Notices AMS, Dec. 2016. To investigate this numerically in MATLAB, try exploring the **Chebfun toolbox**.

The approach we will take is different. We say that if p_1 is the polynomial of degree 1 that interpolates the function f at the points x_0 and x_1 , with $h = x_1 - x_0$, then

$$\max_{x_0 \leq x \leq x_1} |f(x) - p_1(x)| \leq \frac{1}{8} h^2 M_2.$$

So, assuming M_2 is bounded (which is reasonable), we can make p_1 as close to f as we would like by taking a small enough interval $[x_0, x_1]$. The next section of this module is devoted to seeing how this can be used in theory and practice.