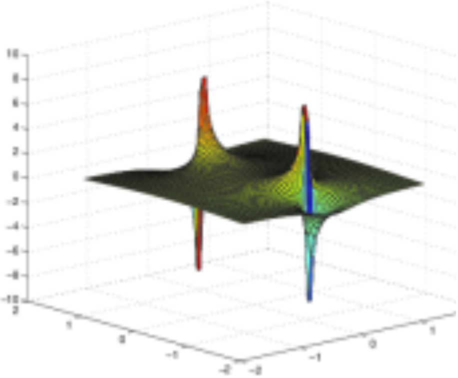
## Annotated slides

§1 Interpolation

# §Wrap-up: Convergence & Runge's Example

MA378/531 – Numerical Analysis II ("NA2")

January 2017



The celebrated Weierstrass approximation theorem states that, given $f$ and a positive number $\varepsilon$, there is a polynomial $p$ such that

$$\max_{x \in [a,b]} |f(x) - p(x)| := \|f - p\|_\infty \leq \varepsilon.$$

Now suppose that $f$ is a continuous function on $[a, b]$ and that $\{p_n\}_{n=0}^{\infty}$ is a sequence of polynomials that interpolate $f$ at $n + 1$ equally spaced points. One might be inclined to believe that

$$\lim_{n \to \infty} \|f - p_n\|_\infty = 0.$$

Another way of thinking about this is recalling the error bound:

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!}|\pi_{n+1}(x)|$$

$\|f^{(n+1)}\|_{\infty,[a,b]}$

we might expect that

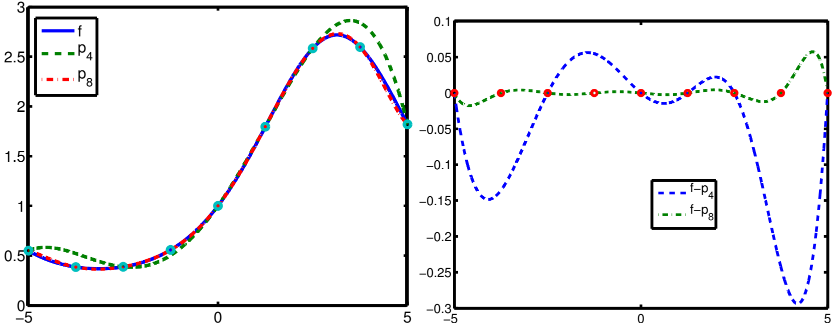
$$\lim_{n\to\infty} \max_{x\in[a,b]} \frac{M_{n+1}}{(n+1)!}|\pi_{n+1}(x)| = 0.$$

In order words, we might think that, in order to find an interpolating polynomial that is as accurate as we would like, we just need to choose large enough $n$.

And **some times** this is true. For example, suppose that $a = -5$, $b = 5$, and $f(x) = e^{\sin(x/2)}$. In Table 1 the errors for successive interpolants are shown.

Table: Errors in polynomial interpolants to $e^{\sin(x/2)}$ on $[-5, 5]$

| $n$ | $\|f - p_n\|_\infty$ |
|---|---|
| 2 | 1.27e-00 |
| 4 | 2.94e-01 |
| 6 | 8.39e-02 |
| 8 | 5.75e-02 |
| 16 | 1.07e-03 |




*Polynomial interpolants to* $e^{\sin(x/2)}$ *on* $[-5,5]$*, and their errors (right)*

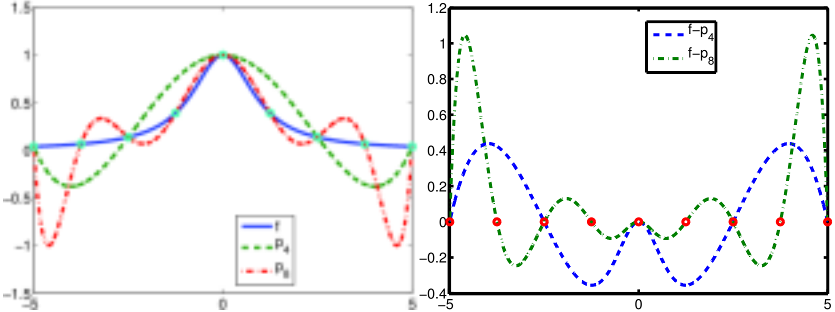
However, there is a famous example of a simple function that cannot be successfully interpolated in this manner

**Runge's Example**

$$f(x) = \frac{1}{1+x^2} \qquad \text{on } [-5, 5].$$

Errors for some $n$ are shown below. Notice they *increase* with $n$.

| $n$ | $\|f - p_n\|$ |
|---|---|
| 2 | 0.65 |
| 4 | 0.44 |
| 6 | 0.62 |
| 8 | 1.05 |
| 16 | 14.39 |
| 20 | 59.66 |
| 22 | 122.91 |
| 24 | 257.21 |



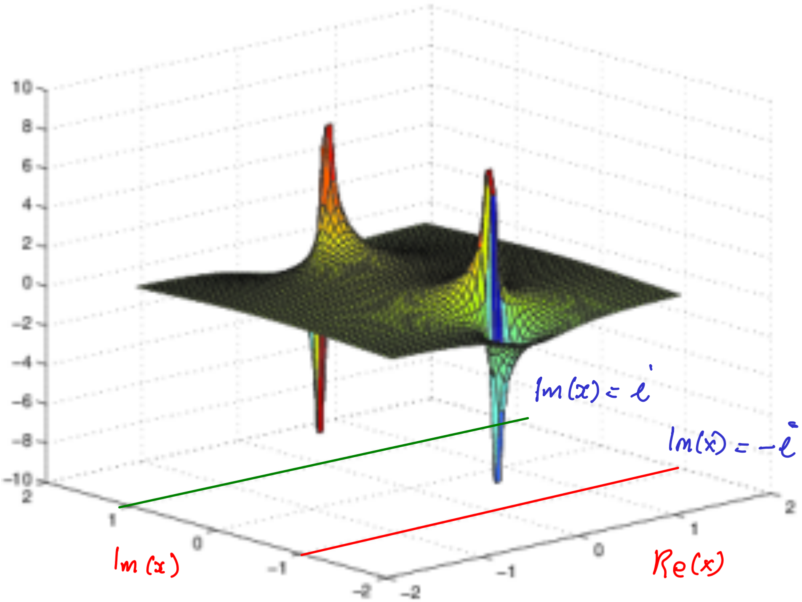

*Polynomial interpolants to* $1/(1+x^2)$ *on* $[-5,5]$

Why is Runge's Example so bad?

The convergence of poly interpolation is related to Taylor Series. It depends on the function, f, being well-behaved... on the region about 0, in the complex plane.

$$f = \frac{1}{1+x^2}$$

is not defined at $x = i$ and $x = -i$.


10
8
6
4
2
0
−2
−4
−6
−8
−10
2
1
0
−1
−2
−2
−1
0
1
2
Im(x) = i
Im(x) = −i
Im(x)
Re(x)

So now it looks like polynomial interpolation is bad, at least on equidistant points. However, our experience in Lab 1 might lead us to be more optimistic: we were able to find a set of points that made the approximation as good we wanted (until round-off error dominated).

Unfortunately, just because we have a good set of points for interpolating one particular function, it does not follow that that set is good for every continuous function: this is ***Faber's Theorem***. This has often led numerical analysts to abandon the idea of interpolation by high-order polynomials completely.

means $f^{(k)}$ defined & continuous.

However, there is a set of points that are useful, if $f$ is smooth enough: the **Chebyshev** points of Lab 1. If you are interested, there read the essay ***Inverse Yogiisms*** by Lloyd N. (Nick) Trefethen, Notices AMS, Dec. 2016. To investigate this numerically in MATLAB, try exploring the Chebfun toolbox.

The approach we will take is different. We say that if $p_1$ is the polynomial of degree 1 that interpolates the function $f$ at the points $x_0$ and $x_1$, with $h = x_1 - x_0$, then

$$\max_{x_0 \le x \le x_1} |f(x) - p_1(x)| \le \frac{1}{8} h^2 M_2.$$

So, assuming $M_2$ is bounded (which is reasonable), we can make $p_1$ as close to $f$ as we would like by taking a small enough interval $[x_0, x_1]$. The next section of this module is devoted to seeing how this can be used in theory and practice.

Finished 30/1/17.