MA378: Class Test, 8 March, 2017: **SOLUTIONS**

Q1. Let l be the **linear spline interpolant** to a function f at the $N+1$ equally spaced points $x_0, x_1, \dots, x_N$. **Use Cauchy's theorem to show that**

$$\max_{x_0 \le x \le x_N} |f(x) - l(x)| \le \frac{h^2}{8} \max_{x_0 \le x \le x_N} |f''(x)|, \qquad (1)$$

where $h = x_i - x_{i-1}$ for each i.

On each subinterval, $[x_{i-1}, x_i]$, $l = l_i$ is just the usual polynomial interpolant to f of degree 1. So, from Cauchy's Theorem,

$f(x) - l_i(x) = \frac{f''(\tau_i)}{2}(x - x_{i-1})(x - x_i)$ for some $\tau_i \in [x_{i-1}, x_i]$

Thus $\max_{x_{i-1} \le x \le x_i} |f(x) - l_i(x)| \le \frac{1}{2} \max_{x_{i-1} \le x \le x_i} |f''(x)| \max_{x_{i-1} \le x \le x_i} |(x - x_{i-1})(x - x_i)|$.

But, the max of $(x - x_{i-1})(x - x_i)$ is at $x = \frac{x_{i-1} + x_i}{2}$. This gives that

$\max_{x_{i-1} \le x \le x_i} |f(x) - l_i(x)| \le \frac{1}{2} \max_{x_{i-1} \le x \le x_i} |f''(x)| \left|\left(\frac{x_i - x_{i-1}}{2}\right)\left(\frac{x_{i-1} - x_i}{2}\right)\right| \le \frac{1}{2} \max_{x_0 \le x \le x_N} \left(\frac{h^2}{4}\right)$.

Since this bound is independent of i, it holds over the whole domain.

Q2. Let $f(x) = 2x^3 - 9x^2 + 9x$ be the function to be interpolated on the interval $[0, 1]$.

(a) Write down the formula for the piecewise **linear** spline, l, and that interpolates this f when $N = 2$ (i.e., the interpolation points are $x_0 = 0, x_1 = 1/2$ and $x_2 = 1$).
What is the value of $l(1/4)$?

(b) Use the formula in (1) to give an upper bound for $\max_{0 \le x \le 1} |f(x) - l(x)|$.

(c) Determine the value of N should one take to ensure that $\max_{0 \le x \le 1} |f(x) - l(x)| \le 10^{-4}$.

(a) The formula for l is

$$l(x) = \begin{cases} l_1(x) & 0 \le x \le 1/2 \\ l_2(x) & 1/2 \le x \le 1 \end{cases},$$

Aside: $f(0) = 0$
$f(1/2) = \frac{2}{8} - \frac{9}{4} + \frac{9}{2} = 5/2$
$f(1) = 2$

where $l_1(x) = f(0)\frac{(1/2 - x)}{1/2} + f(1/2)\frac{x}{1/2} = 5x$

and

$l_2(x) = f(1/2)\frac{(1-x)}{1/2} + f(1)\frac{(x - 1/2)}{1/2} = 5(1-x) + 4(x - 1/2) = -x + 3$

$l(1/4) = l_1(1/4) = 5/4$,

(b). Since $h = 1/2$, the bound in (1) gives $|f(x) - l(x)| \le \frac{1}{32} \max_{0 \le x \le 1} |f''(x)|$.

Here $f'(x) = 6x^2 - 18x + 9$, and $f''(x) = 12x - 18$.

Therefore $\max_{0 \le x \le 1} |f''(x)| = |f''(0)| = 18$. So $|f(x) - l(x)| \le 18/32 \simeq \boxed{0.5625}$

(c) We need to choose h so that $|f(x) - l(x)| \le \frac{h^2}{8}(18) \le 10^{-4}$.

Therefore $h^2 \le \left(\frac{8}{18}\right) \times 10^{-4} = 4.444 \times 10^{-5}$, giving $h \le 0.00666$.

Since $h = 1/N$, we need $N \ge 1/0.00666 = \underline{150}$.