Started 1/2/17.

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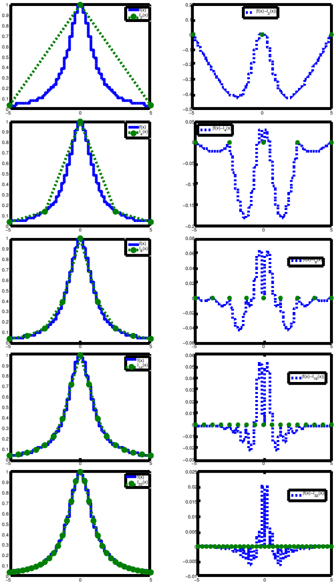
§2 Piecewise Polynomial Interpolation

§2.1 Linear Interpolating Splines

MA378/531 – Numerical Analysis II ("NA2")

February 2017



In Section 1.6 and in Lab 1, we learned that it is not always a good idea to interpolate functions by a high-order polynomials at equally spaced points. However, it transpires that it is possible to obtain a very good approximations using a very simple method. The trick is to use a **spline**: a ***piecewise polynomial interpolating function***.

We'll consider three important example of splines:

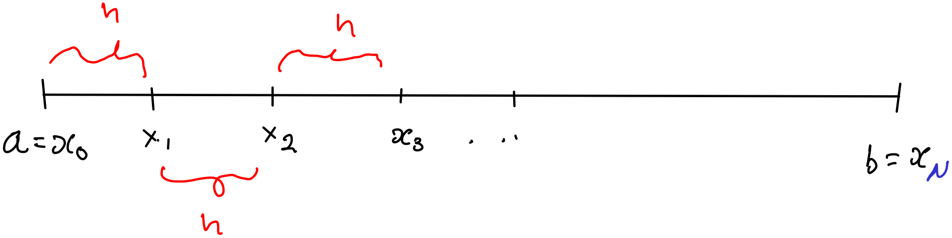
1. ***linear splines***
2. ***(natural) cubic splines***.
3. ***Hermite*** piecewise cubics.

For more details about splines, have a look at Chap. 11 of Süli and Mayers, and Lectures 10 and 11 Stewart's "Afternotes goes to Grad School".

In this section, we always have n equally spaced points: let $h = (b-a)/n$, then

$$a = x_0, \quad b = x_n \quad \text{and} \quad x_i = x_0 + ih.$$

Often these are referred to as *knots points* (or simply as *knots*), and denote the set of knot points by $\omega^N := \{x_i\}_{i=0}^N$.



We first study the *piecewise linear interpolant*, also called a *linear spline*. We will see that they have important properties, including

(a) they are easy to construct and analyse;

(b) the bound on the error decreases as the number of interpolation points increases;

today

(c) the error we get using a linear spline is no more than twice the error using the best possible (piecewise linear) approximation; amd

(d) of all the interpolants to f at a given set of points, the linear spline is the one with the smallest 1st derivative.

"Minimum Energy property".

Definition

Let f be a function that is continuous on $[a, b]$. The *linear spline interpolant* to f is the continuous function l such that

(i) $l(x_i) = f(x_i)$ for each $i = 0, 1, \dots, n$, (interpolates f).

(ii) l is a linear function l_i on each interval $[x_{i-1}, x_i]$. That is,

$$l(x) = \begin{cases} l_1(x) & x_0 \le x \le x_1 \\ l_2(x) & x_1 \le x \le x_2 \\ \dots & \\ l_n(x) & x_{n-1} \le x \le x_n \end{cases}$$

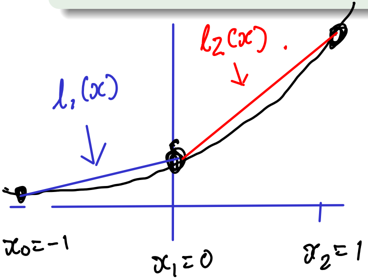
piece-wise $\Rightarrow$ "different formula on each subinterval".

It is easy to write down a formula for the l_i, based on Lagrange polynomials. Set $h = (b - a)/n$. Then, for $x \in [x_{i-1}, x_i]$

$$l_i(x) = f(x_{i-1}) \frac{x_i - x}{h} + f(x_i) \frac{x - x_{i-1}}{h}. \qquad (1)$$

Example

Write down the linear spline interpolant to $f(x) = e^x$ at the knot points $\{-1, 0, 1\}$.



$$l_1(x) = f(x_0)\frac{x_1 - x}{h} + f(x_1)\frac{x - x_0}{h}.$$

$$= e^{-1}(-x) + (x+1).$$

$$l_2(x) = f(x_1)\frac{x_2 - x}{h} + f_2\frac{x - x_1}{h}$$

$$= (1 - x) + e x.$$

Then

$$l(x) = \begin{cases} -e^{-1}x + x+1 & -1 \le x < 0 \\ 1 - x + e x & 0 \le x \le 1. \end{cases}$$

We know that if p_n is the polynomial of degree n that interpolates f at n equally spaced points, it does **not** follow that $p_n \to f$ as $n \to \infty$. But as we will see, the piecewise linear interpolant to f converges to f, albeit slowly.

This is verified in the following theorem, which is a direct consequence of Cauchy's theorem.

Theorem

Suppose that f, f' and f'' are all continuous and defined on the interval $[a, b]$. Let l be the linear spline interpolant to f on the $n+1$ equally spaced points $a = x_0 < x_1 \cdots < x_n = b$ with $h = x_i - x_{i-1} = (b-a)/n$. Then

$$\|f - l\|_\infty \le \frac{h^2}{8}\|f''\|_\infty,$$

(Here, as usual, $\|g\|_\infty$ is defined as $\max_{a \le x \le b} |g(x)|$.)

First, Cauchy's Theorem with $n=2$ gives that,

$$|f(x) - p_1(x)| \leq \frac{|f''(\tau)|}{2} |(x-x_0)(x-x_1)|$$

So, for any $x \in [x_{i-1}, x_i]$,

$$|f(x) - l_i(x)| \leq \frac{|f''(\tau_i)|}{2} |(x-x_{i-1})(x-x_i)|$$

where $\tau_i \in [x_{i-1}, x_i]$.

Then

$$\max_{x_{i-1} \leq x \leq x_i} |f(x) - l_i(x)| \leq \frac{1}{2}\left(\max_{x_{i-1} \leq x \leq x_i} |f''(x)|\right) \max_{x_{i-1} \leq x \leq x_i} |(x-x_{i-1})(x-x_i)|$$

See board.

It follows directly from this theorem that

$$\lim_{n\to\infty} \|f - l\|_\infty = 0.$$

Because $h = \frac{b-a}{n}$ so, as $n \to \infty$, $h \to 0$.

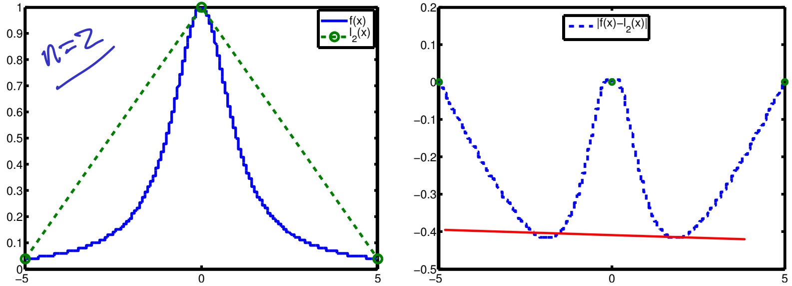
So $\frac{h^2}{8} \|f''\|_\infty \to 0$.

Example

The figure below shows linear spline interpolations of Runge's example:

$$f(x) = \frac{1}{1+x^2} \text{ on } [-5, 5].$$

These diagrams appear to support our assertion that the error tends to zero as $n \to \infty$.



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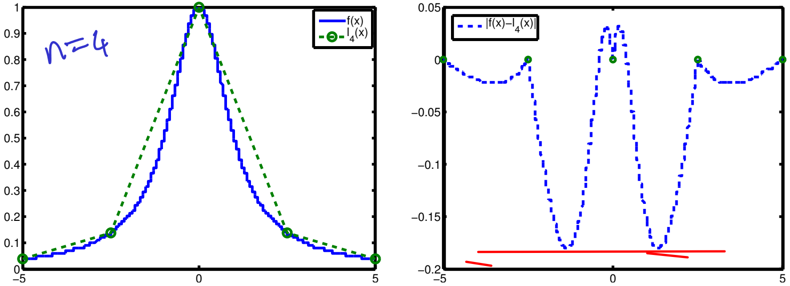

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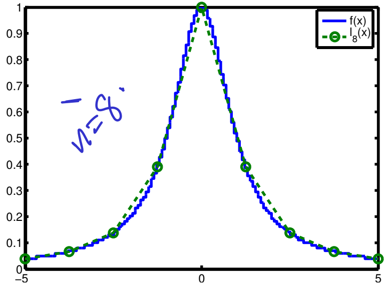

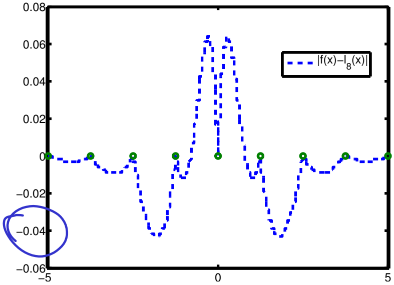

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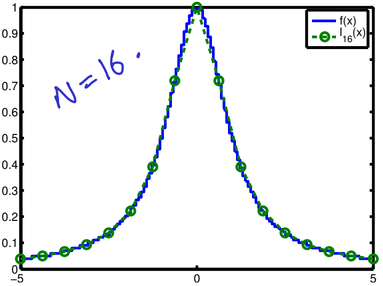

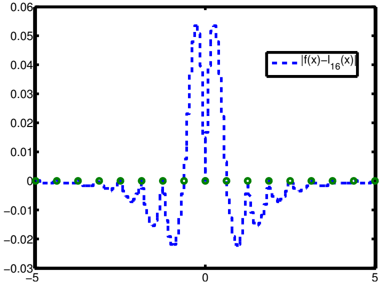

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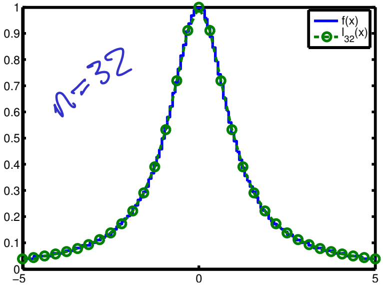
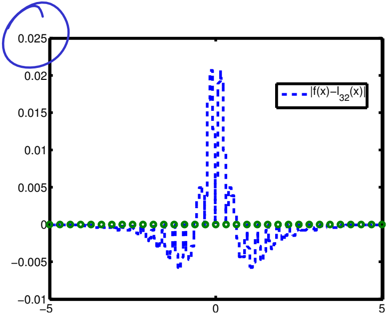




Example

Suppose you are interpolating $f(x) = e^x$ on ~~n~~ N equally spaced intervals between $x_0 = -1$ and $x_{\cancel{n}\,N} = 1$. What value of n would you have to take to ensure that the maximum error is less than 10^{-2}?

We want $|f(x) - \ell(x)| \leq 10^{-2}$ for all $x \in [-1, 1]$.

So choose N such that

$$\|f - \ell\|_\infty \leq \underbrace{\frac{h^2}{8}\|f''\|}_{\text{we can control}} \leq 10^{-2}.$$

$f(x) = e^x$, so $f''(x) = e^x$, so $\|f\|_\infty = e^1 = 2.7183$.

We require h so that

$$\frac{h^2}{8}(2.7183) \leq 10^{-2} \Rightarrow h^2 \leq \frac{(8)(0.01)}{2.7183} = 0.02943.$$

So $h \leq 0.17155$. Next, $h = \dfrac{b-a}{N} = \dfrac{2}{N}$. So need

$$\frac{2}{N} \leq 0.17155 \Rightarrow N \geq 11.655$$

So take $N = 12$.

For the next part of the analysis it will help to think of piecewise linear interpolation as an *operator*. Then we can compare the linear spline to all the other piecewise linear approximations.

First, observe that one can define an infinite number of piecewise linear functions on a given set of knot points ω^N. We'll call the set of these functions $\mathcal{L}$.

Definition

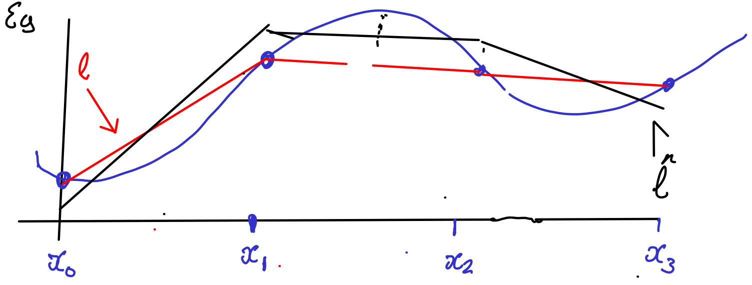
For a fixed set of knot points ω^N, let L be the operator that maps the continuous function f to its linear spline interpolant $l \in \mathcal{L}$.

Now suppose that $g \in \mathcal{L}$. Then $\mathrm{L}(g) = g$. That is L is a *projection*: $\mathrm{L}(\mathrm{L}(f)) = \mathrm{L}(f)$.

It is not hard to see that one could find a different function $\hat{l} \in \mathcal{L}$ that is a better approximation of f in sense that

$$\max_{x_0 \leq x \leq x_n} |f(x) - \hat{l}(x)| < \max_{x_0 \leq x \leq x_n} |f(x) - l(x)|.$$

However, l is very easy to find, and the associated error is no worse than twice $\|f(x) - \hat{l}(x)\|_\infty$.



Theorem (Stewart's "Afternotes goes to grad school", Lecture 10)

Let $l = \mathrm{L}(f)$. For all $\hat{l} \in \mathcal{L}$,

$$\|f - l\|_\infty \le 2\|f - \hat{l}\|_\infty.$$

(That L is a projection is key to the proof.)

Proof

$$\|f - l\|_\infty = \|f - \hat{l} + \hat{l} - l\|_\infty$$

$$\le \|f - \hat{l}\|_\infty + \|\hat{l} - l\|_\infty \quad \text{(by the triangle inequality)}$$

$$= \|f - \hat{l}\|_\infty + \|L(\hat{l}) - L(f)\|_\infty \quad (L \text{ is a projection on } \mathcal{L})$$

$$= \|f - \hat{l}\|_\infty + \|L(\hat{l} - f)\|_\infty \quad (L \text{ is linear}).$$

$$\le \|f - \hat{l}\|_\infty + \|\hat{l} - f\|_\infty,$$

as required.

$$\left(\|L(f)\|_\infty = \max_i |l(x_i)| = \max_i |f(x_i)| \le \|f\|_\infty \right)$$

The final interesting property of l that we will study is called the *minimum energy property*.

Definition

Let u be a function that is continuous and defined on the interval $[a, b]$ except, maybe, at the (countable set) ω^N of knot points[a] Then the 2-norm of u is

$$\|u\|_{2,[a,b]} := \Big(\int_a^b u^2(x)dx\Big)^{1/2}.$$

Usually we just write this as $\|u\|_2$.

[a] More precisely, we should say "everywhere, except on a set of measure zero". However, since some of you are not taking those courses until next year, we'll air-brush over the exact definition.

Let H^1 be the set of all functions u that are continuous on $[a, b]$ and have $\|u'\|_2 < \infty$. Note that $l' \in H^1$, even though we have not properly defined l' at the mesh points ω^N.

ie u' is integrable.

$\int u\,dv = uv - \int v\,du$

Theorem (Süli and Mayers, Thm. 11.2)

Let w be any function in H^1 that interpolates the function f at the points in ω^N. Let l we the linear spline interpolant of f. Then

$$\|l'\|_2 \le \|w'\|_2.$$

Proof: $\|w'\|_2^2 = \int_a^b (w')^2\,dx = \int_a^b ((w' - l') + l')^2\,dx.$

$$= \underbrace{\int_a^b (w'-l')^2\,dx}_{\|w'-l\|_2^2 \geq 0} + \underbrace{\int_a^b (l')^2\,dx}_{\|l'\|_2^2} + \underbrace{2\int_a^b l'(w'-l')\,dx}_{=0\text{, as we now show.}}$$

To see this, $\int_a^b l'(w'-l')\,dx = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} l'(w'-l')\,dx.$

But
$$\int_{x_i}^{x_{i+1}} \underbrace{l'}_{u} \underbrace{(w-l)'\,dx}_{dv} = l'\underbrace{(w-l)}_{w(x_i)=l(x_i)=f(x_i)}\Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} (w-l)\underbrace{l''}_{0}\,dx$$

l is p.w. linear