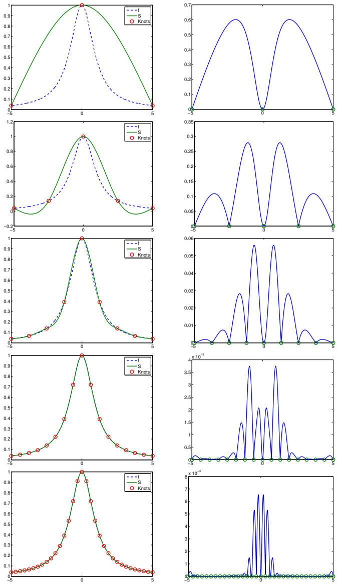


§2 Piecewise Polynomial Interpolation

§2.2 (Natural) Cubic Splines

MA378/531 – Numerical Analysis II (“NA2”)

February 2017

The **Cubic Spline**[1] is perhaps the most useful and popular interpolating function used in numerical analysis, automotive and aeronautical engineering, computer and film animation, digital photography, financial modelling, and as many other areas as there are human endeavours where continuous processes are modelled by discrete ones.

[1] It is generally accepted that (mathematical) splines were first described by Isaac Schoenberg (1903-1990) during WW2. However their physical realisation had been used in the ship-building and aircraft industries prior to that.

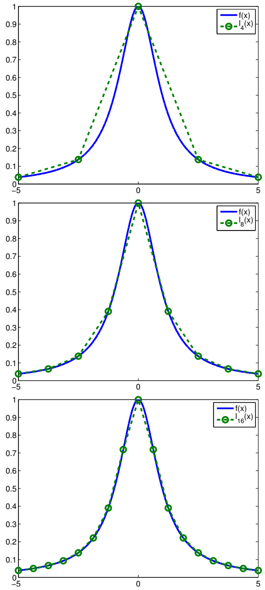
One could argue that the most fundamental short-coming of the linear spline interpolant to f is that they are not "***smooth***": that is, although

$$l_i(x_i) = l_{i+1}(x_i),$$

it is generally the case that

$$l_i'(x_i) \neq l_{i+1}'(x_i).$$

Also, the interpolating function cannot capture the "***curvature***" of f. (If you think about this last statement, you'll see that it can be expressed as $l''(x) = 0$ for all $x \in [a, b]$.)



Cubic splines try to balance the simplicity of the linear spline approach — by using a low-order polynomial on each interval — with the desire for curvature and more smoothness – by using cubics, and by forcing adjacent ones, and their first and second derivatives, to agree at knot points.

Definition (Cubic Spline)

Let f be a function that is continuous on $[a, b]$. The *cubic spline interpolant* to f is the continuous function S such that

(i) for $i = 1, \dots, N$, on each interval $[x_{i-1}, x_i]$ let $S(x) = s_i(x)$, where each of the s_i is a cubic polynomial.

(ii) $s_i(x_{i-1}) = f(x_{i-1})$ for $i = 1, \dots, N$,

(iii) $s_i(x_i) = f(x_i)$ for $i = 1, \dots, N$,

(iv) $s_i'(x_i) = s_{i+1}'(x_i)$ for $i = 1, \dots, N-1$,

(v) $s_i''(x_i) = s_{i+1}''(x_i)$ for $i = 1, \dots, N-1$.

So, we have defined the cubic spline S as a function that interpolates f at $N+1$ points, has continuous first and second derivatives on $[x_0, x_N]$ and is a cubic polynomial on each of the n intervals $[x_{i-1}, x_i]$. That is, it is ***piecewise cubic***.

We know that one can write a cubic as $a_0 + a_1x + a_2x^2 + a_3x^3$. So it takes 4 terms to uniquely define a single cubic. To define the spline we need $4N$ terms. They can be found by solving $4N$ (linearly independent) equations. But the definition only gives $4N - 2$ equations.

The "missing"equations can be chosen in a number of ways:

(i) by setting $S''(x_0) = 0$ and $S''(x_N) = 0$. This is called a *natural* spline, and is the approach we'll take.

(ii) by setting $S'(x_0) = 0$ and $S'(x_N) = 0$. This is called a *clamped* spline.

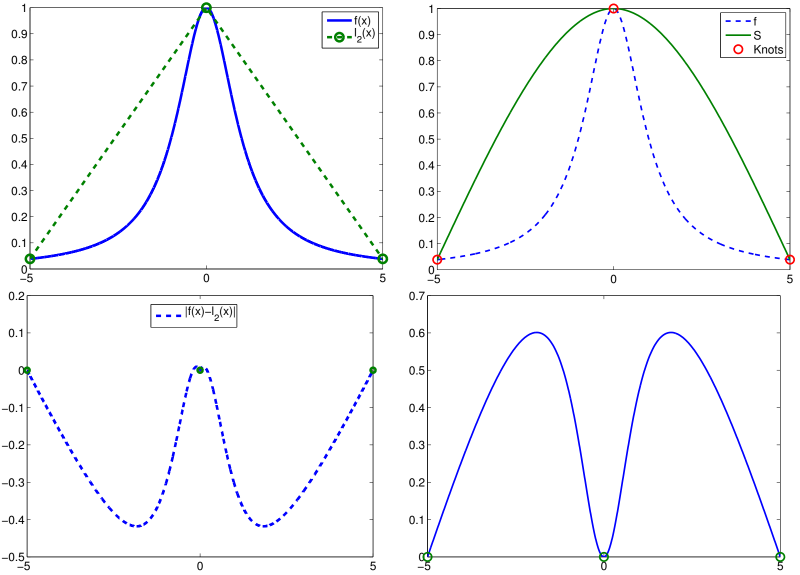
(iii) set $S'(x_0) = S'(x_N)$ and $S''(x_0) = S''(x_N)$. This is the *periodic* spline and is used for interpolating, say, trigonometric functions.

(iv) only use $N - 2$ components of the spline: $s_2, \ldots, s_{N-1}$. But extend to two end ones so that $s_2(x_0) = f(x_0)$ and $s_{N-1} = f(x_N)$. This is called the *not-a-knot* condition.

The following figures show the **linear spline** (left) and ***natural cubic spline*** (right), and errors, interpolants to $f(x) = 1/(1+x^2)$ (with $a=-5$ and $b=5$ as usual) for various N.

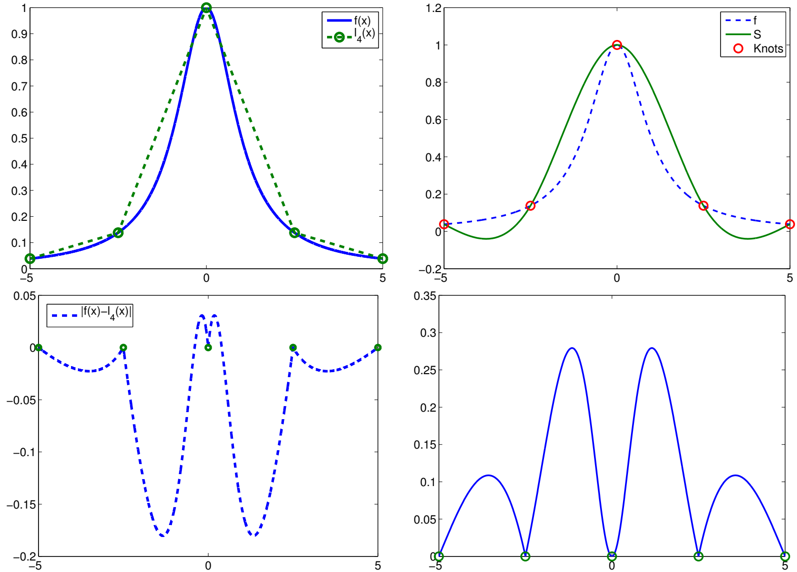
Compared with linear splines, we seem to get significantly better approximation. Moreover, the rate at which the error decreases with respect to h is much faster.

$N = 2$



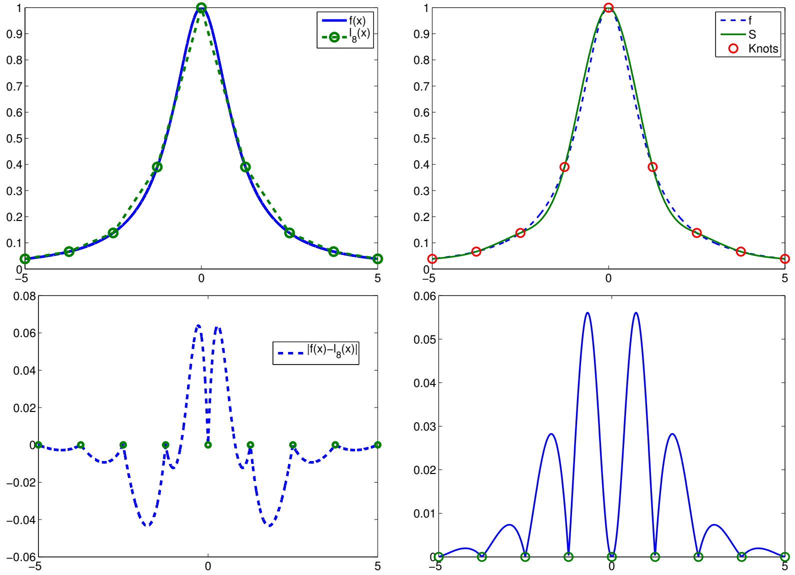
Definition

$N = 4$

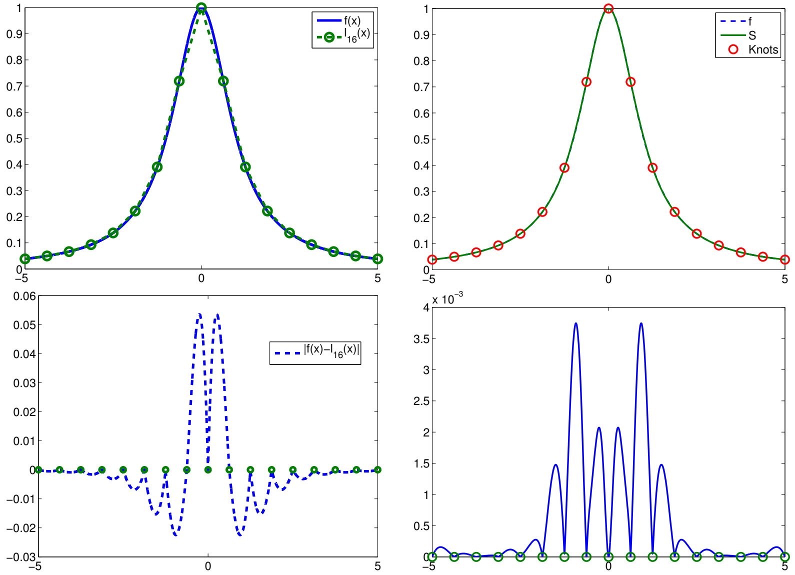


Definition

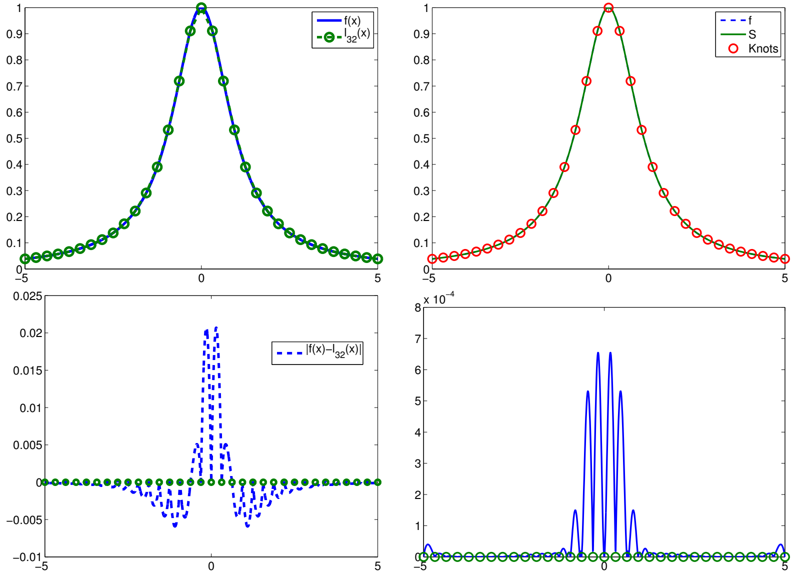
$N = 8$



$$N = 16$$



$$N = 32$$



Some notation:

- $h = x_i - x_{i-1} = (x_N - x_0)/N$ for all i.
- $f_i := f(x_i)$.

To construct the spline, first observe that if S is piecewise cubic, then S' is piecewise quadratic, and S'' is piecewise linear.

(i) Let $\sigma_i = S''(x_i)$ for $i = 0, \dots, N$.

(ii) Integrate twice:

We get

$$s_i(x) = \alpha_i(x - x_{i-1}) + \beta_i(x_i - x) + \frac{\sigma_{i-1}}{6h}(x_i - x)^3 + \frac{\sigma_i}{6h}(x - x_{i-1})^3, \quad (1)$$

for $x \in [x_{i-1}, x_i]$.

(iii) The α_i and β_i arose as the constants of integration. To find them:

and so, for $i = 1, 2, \ldots, N$

$$\alpha_i = \frac{f_i}{h} - \frac{h}{6}\sigma_i, \quad \beta_i = \frac{f_{i-1}}{h} - \frac{h}{6}\sigma_{i-1}. \tag{2}$$

Notice that this gives $\beta_i = \alpha_{i-1}$.

(iv) So, now we "just" need the equations for σ_i. Two of these come from the fact that this is a "natural" cubic spline, with $S''(x_0) = 0$ and $S''(x_N) = 0$, therefore $\sigma_0 = 0$ and $\sigma_N = 0$. The remaining $N-1$ equations come from the fact that S' is continuous at $x_1, x_2, \dots, x_{n-1}$. So we set that

$$s_i'(x_i) = s_{i+1}'(x_i).$$

After some work (see exercises...), we can show this means that the system is:

$$\sigma_0 = 0, \tag{3a}$$

$$\frac{1}{6}(\sigma_{i-1} + 4\sigma_i + \sigma_{i+1}) = \frac{1}{h^2}(f_{i-1} - 2f_i + f_{i+1}) \tag{3b}$$

for $i = 1, \dots, N-1$,

$$\sigma_N = 0. \tag{3c}$$

Example

Find the natural cubic spline interpolant to f at the points $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$ where $f_0 = 0, f_1 = 2, f_2 = 1$ and $f_3 = 0$.

[We won't do all the details in class, so here they are]

Solution:

From (1) we see we are looking for $S(x)=$

$$\begin{aligned} \alpha_1 x + \beta_1(1-x) + \frac{\sigma_0}{6h}(1-x)^3 + \frac{\sigma_1}{6h}x^3, &x \in [0,1], \\ \alpha_2(x-1) + \beta_2(2-x) + \frac{\sigma_1}{6h}(2-x)^3 + \frac{\sigma_2}{6h}(x-1)^3, &x \in [1,2], \\ \alpha_3(x-x_2) + \beta_3(3-x) + \frac{\sigma_2}{6h}(3-x)^3 + \frac{\sigma_3}{6h}(x-2)^3, &x \in [2,3]. \end{aligned}$$

We first solve for the σ_i using (3):

$$\frac{1}{6}\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 4 & 1 & 0\\ 0 & 1 & 4 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sigma_0\\ \sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix} = \begin{pmatrix}0\\ -18\\ 0\\ 0\end{pmatrix}$$

This gives

$$\sigma_0 = 0, \quad \sigma_1 = -\frac{24}{5}, \quad \sigma_2 = \frac{6}{5}, \quad \sigma_3 = 0.$$

Now use (2) to get

$$\alpha_1 = \frac{14}{5}, \quad \alpha_2 = \frac{4}{5}, \quad \alpha_3 = 0.$$

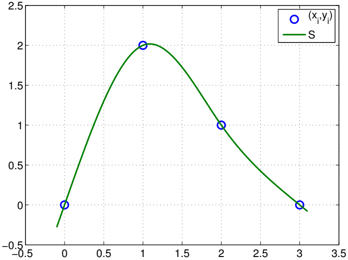
and

$$\beta_1 = 0 \quad \beta_2 = \frac{14}{5}, \quad \beta_3 = \frac{4}{5}.$$

The answer is

$$S(x) = \begin{cases} \frac{14}{5}x - \frac{4}{5}x^3, & x \in [0,1], \\ \frac{4}{5}(x-1) + \frac{14}{5}(2-x) - \\ \qquad \frac{4}{5}(2-x)^3 + \frac{1}{5}(x-1)^3, & x \in [1,2], \\ \frac{4}{5}(3-x) + \frac{1}{5}(3-x)^3. & x \in [2,3]. \end{cases}$$

This is shown below.



We state the following error estimates without proof. You don't need to know these, or be able to prove them. But you should be able to use them if required.

Theorem

If $f \in C^4[a,b]$ and S is its cubic spline interpolant on $N+1$ equally spaced points, then

$$\|f - S\|_\infty \le \frac{5}{384} M_4 h^4,$$
$$\|f' - S'\|_\infty \le \frac{1}{24} M_4 h^3,$$
$$\|f'' - S''\|_\infty \le \frac{3}{8} M_4 h^2,$$

where $M_4 = \|f^{(iv)}\|_\infty$.

Example

Give an upper bound on the error for the cubic spline interpolant to $f = e^x$ on the interval $[-1, 1]$ with $N = 10$ mesh points.

Example

What is the smallest value of N that you can take which ensures that, if interpolating $f(x) = e^x$ on N equally sized intervals on $[-1, 1]$, the error is less that 10^{-8}? How does this compare with a linear spline interpolation (see corresponding example for linear splines).

Recall the minimum energy property of linear splines. There is an analogous result for natural cubic splines.

Theorem

Let u be any function that interpolates f at ω^N, and is such that $u \in H^2(x_0, x_N)$. Then

$$\|S''\|_2 \leq \|u''\|_2.$$

The proof is not hard - it is analogous to the proof of Theorem 2.1.9.