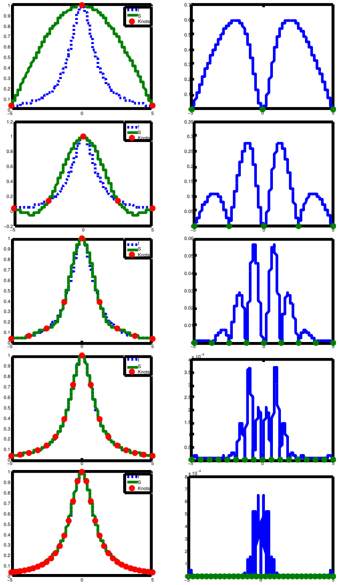


§2 Piecewise Polynomial Interpolation

§2.2 (Natural) Cubic Splines

MA378/531 – Numerical Analysis II ("NA2")

February 2017

Started 8/2/17

The **Cubic Spline**[1] is perhaps the most useful and popular interpolating function used in numerical analysis, automotive and aeronautical engineering, computer and film animation, digital photography, financial modelling, and as many other areas as there are human endeavours where continuous processes are modelled by discrete ones.

[1] It is generally accepted that (mathematical) splines were first described by Isaac Schoenberg (1903-1990) during WW2. However their physical realisation had been used in the ship-building and aircraft industries prior to that.

One could argue that the most fundamental short-coming of the linear spline interpolant to f is that they are

(1) not "***smooth***": that is, although

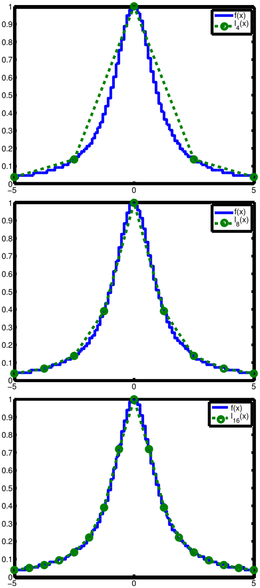
$$l_i(x_i) = l_{i+1}(x_i),$$

l has "corners"

it is generally the case that

$$l'_i(x_i) \neq l'_{i+1}(x_i).$$

(ii) Also, the interpolating function cannot capture the "***curvature***" of f. (If you think about this last statement, you'll see that it can be expressed as $l''(x) = 0$ for all $x \in [a, b]$.)



Definition

Cubic splines try to balance the simplicity of the linear spline approach — by using a low-order polynomial on each interval — with the desire for curvature and more smoothness – by using cubics, and by forcing adjacent ones, and their first and second derivatives, to agree at knot points.

Definition (Cubic Spline)

Let f be a function that is continuous on $[a, b]$. The *cubic spline interpolant* to f is the continuous function S such that

(i) for $i = 1, \ldots, N$, on each interval $[x_{i-1}, x_i]$ let $S(x) = s_i(x)$, where each of the s_i is a cubic polynomial. } Piece-wise cubic

(ii) $s_i(x_{i-1}) = f(x_{i-1})$ for $i = 1, \ldots, N$,

(iii) $s_i(x_i) = f(x_i)$ for $i = 1, \ldots, N$, } s_i interpolates f at x_{i-1} & x_i

(iv) $s_i'(x_i) = s_{i+1}'(x_i)$ for $i = 1, \ldots, N-1$, } S' is continuous.

(v) $s_i''(x_i) = s_{i+1}''(x_i)$ for $i = 1, \ldots, N-1$ } S'' is continuous.

So, we have defined the cubic spline S as a function that interpolates f at $N+1$ points, has continuous first and second derivatives on $[x_0, x_N]$ and is a cubic polynomial on each of the n intervals $[x_{i-1}, x_i]$. That is, it is ***piecewise cubic***.

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(iv) $s_i'(x_i) = s_{i+1}'(x_i)$ for $i = 1, \ldots, N-1$, $\rightarrow$ N-1 equations

(v) $s_i''(x_i) = s_{i+1}''(x_i)$ for $i = 1, \ldots, N-1$. $\rightarrow$ N-1 equations

4N-2

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We know that one can write a cubic as $a_0 + a_1x + a_2x^2 + a_3x^3$. So it takes 4 terms to uniquely define a single cubic. To define the spline we need $4N$ terms. They can be found by solving $4N$ (linearly independent) equations. But the definition only gives $4N - 2$ equations.

The "missing"equations can be chosen in a number of ways:

(i) by setting $S''(x_0) = 0$ and $S''(x_N) = 0$. This is called a *natural* spline, and is the approach we'll take.

(ii) by setting $S'(x_0) = 0$ and $S'(x_N) = 0$. This is called a *clamped* spline.

(iii) set $S'(x_0) = S'(x_N)$ and $S''(x_0) = S''(x_N)$. This is the *periodic* spline and is used for interpolating, say, trigonometric functions.

(iv) only use $N - 2$ components of the spline: $s_2, \ldots, s_{N-1}$. But extend to two end ones so that $s_2(x_0) = f(x_0)$ and $s_{N-1} = f(x_N)$. This is called the *not-a-knot* condition.

The following figures show the **linear spline** (left) and ***natural cubic spline*** (right), and errors, interpolants to $f(x) = 1/(1 + x^2)$ (with $a = -5$ and $b = 5$ as usual) for various N.

Compared with linear splines, we seem to get significantly better approximation. Moreover, the rate at which the error decreases with respect to h is much faster.

Definition

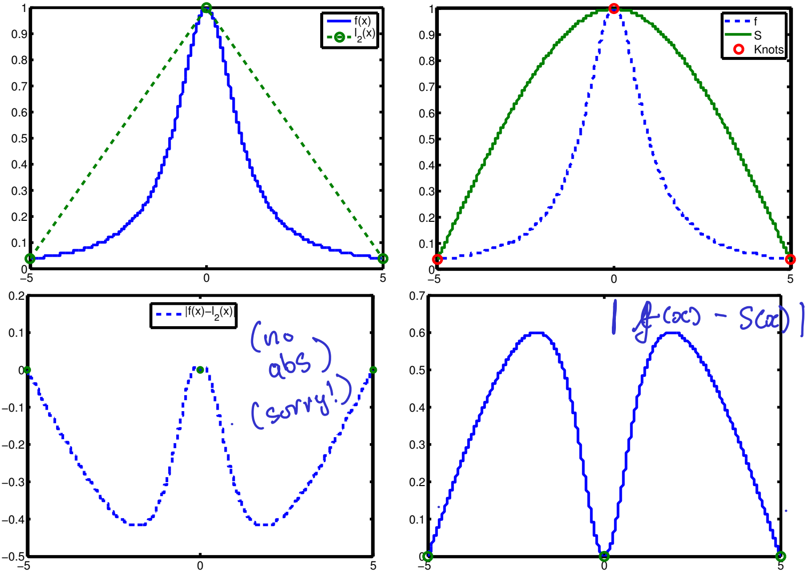
Linear

$N = 2$

Cubic spline

Interpolants

Errors.

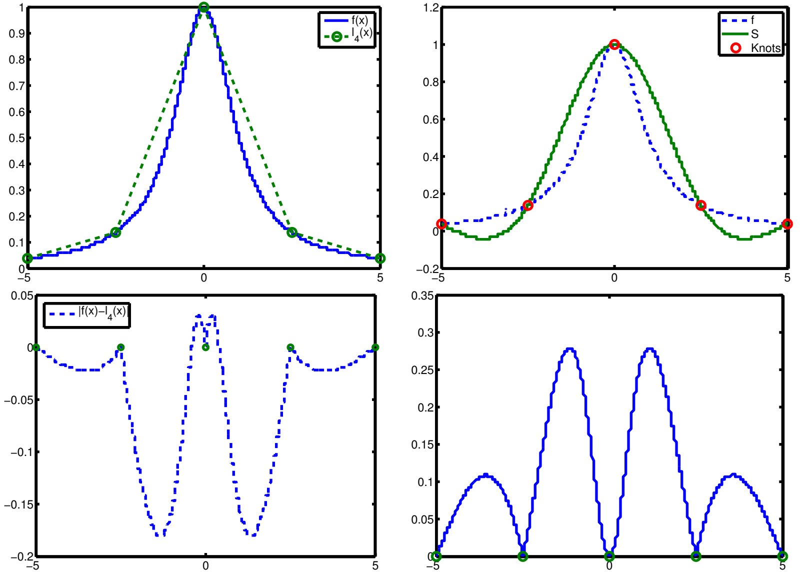


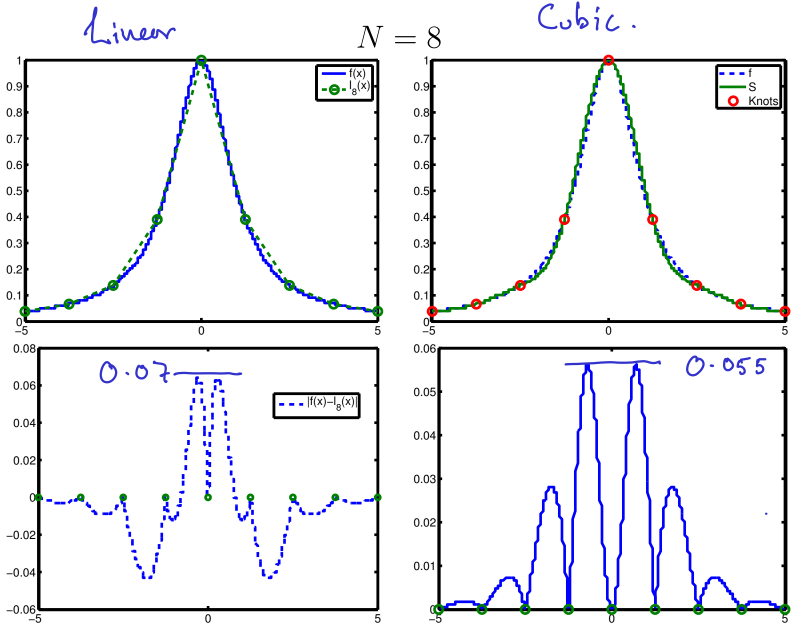
Definition

linear

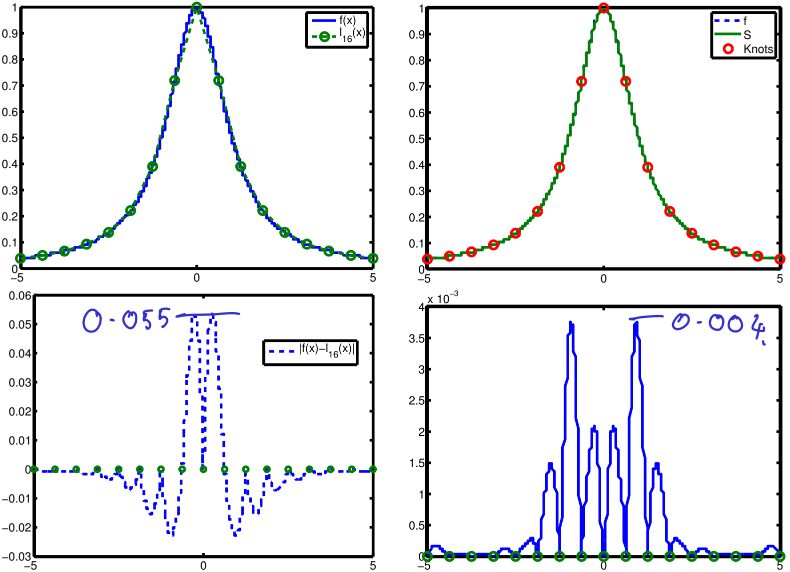
$N = 4$

Cubic.




Linear
Cubic.
$N = 8$
f(x)
$l_8(x)$
f
S
Knots
0.07
$|f(x)-l_8(x)|$
0.055

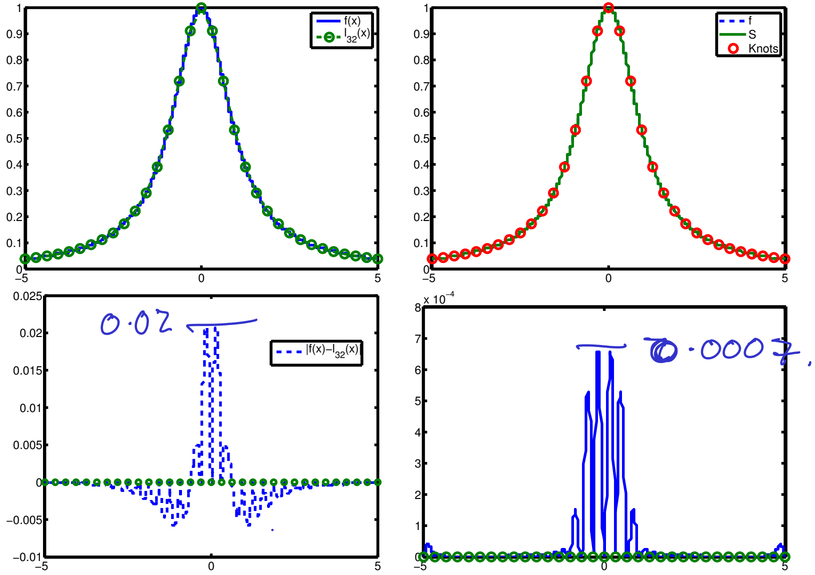
Linear $N = 16$ Cubic.



Linear

Cubic.

$N = 32$



Some notation:

- $h = x_i - x_{i-1} = (x_N - x_0)/\cancel{n}N$ for all i.
- $f_i := f(x_i)$. ← short hand

write $f(x_1)$ as f_1

To construct the spline, first observe that if S is piecewise cubic, then S' is piecewise quadratic, and S'' is piecewise linear.

(i) Let $\boxed{\sigma_i = S''(x_i)}$ for $i = 0, \ldots, N$.

Since S_i is a cubic polynomial, S_i' is quadratic, and S_i'' is linear (ie S'' is piecewise linear). Using the formula for linear splines we can write

$$S_i''(x) = \sigma_{i-1}\frac{x_i - x}{h} + \sigma_i \frac{x - x_{i-1}}{h}.$$

where the σ_i are, as yet, unknown.

Constant of integration

(ii) Integrate twice:

$$s_i'(x) = -\sigma_{i-1}\frac{(x_i-x)^2}{2h} + \sigma_i\frac{(x-x_{i-1})^2}{2h} + A_i$$

And again:

$$s_i(x) = \sigma_{i-1}\frac{(x_i-x)^3}{6h} + \sigma_i\frac{(x-x_{i-1})^3}{6h} + A_i x + B_i.$$

The A_i & B_i are arbitrary constants. So we can express them in terms of arbitrary α_i, β_i:

$$A_i x + B_i = \alpha_i(x - x_{i-1}) + \beta_i(x_i - x)$$

(That is: $\alpha_i + \beta_i = A_i$, $-\alpha_i x_{i-1} + \beta_i x_i = B_i$).

We get

$$s_i(x) = \alpha_i(x - x_{i-1}) + \beta_i(x_i - x) + \frac{\sigma_{i-1}}{6h}(x_i - x)^3 + \frac{\sigma_i}{6h}(x - x_{i-1})^3, \quad (1)$$

for $x \in [x_{i-1}, x_i]$.

Finished here 8/2/17.

(iii) The α_i and β_i arose as the constants of integration. To find them: use that $s_i(x_{i-1}) = f_{i-1}$, $s_i(x_i) = f_i$.

That is,

$$s_i(x_{i-1}) = \beta_i h + \frac{\sigma_{i-1}}{6h} h^3 = f_{i-1} \Rightarrow \boxed{\beta_i h + \frac{\sigma_{i-1} h^2}{6} = f_{i-1}}$$

Similarly

$$s_i(x_i) = \alpha_i h + \frac{\sigma_i h^2}{6} = f_i$$

We know f_{i-1} & f_i. If we knew σ_{i-1} and σ_i, we can solve for α_i & β_i.

and so, for $i = 1, 2, \dots, N$

$$\alpha_i = \frac{f_i}{h} - \frac{h}{6}\sigma_i, \quad \beta_i = \frac{f_{i-1}}{h} - \frac{h}{6}\sigma_{i-1}. \qquad (2)$$

{Note that $\beta_i = \alpha_{i-1}$.}

Constructing Cubic Splines

(iv) So, now we "just" need the equations for σ_i. Two of these come from the fact that this is a "natural" cubic spline, with $S''(x_0) = 0$ and $S''(x_N) = 0$, therefore $\sigma_0 = 0$ and $\sigma_N = 0$. The remaining $N-1$ equations come from the fact that S' is continuous at $x_1, x_2, \dots, x_{n-1}$. So we set that

First derivative of s_i & s_{i+1} are equal at x_i

$$s_i'(x_i) = s_{i+1}'(x_i).$$

After some work (see exercises...), we can show this means that the system is:

$$\sigma_0 = 0, \tag{3a}$$

$$\frac{1}{6}\underbrace{(\sigma_{i-1} + 4\sigma_i + \sigma_{i+1})} = \frac{1}{h^2}\underbrace{(f_{i-1} - 2f_i + f_{i+1})} \tag{3b}$$

for $i = 1, \dots, N-1$,

$$\sigma_N = 0. \tag{3c}$$

We'll see this again – Simpson's Rule

approximation for $f''(x_i)$ ("finite difference").

Example

Find the natural cubic spline interpolant to f at the points $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$ where $f_0 = 0, f_1 = 2, f_2 = 1$ and $f_3 = 0$.

[We won't do all the details in class, so here they are]

Solution:

From (1) we see we are looking for $S(x)$=

$$\begin{aligned}
S_1(x) = {} & \alpha_1 x + \beta_1(1-x) + \frac{\sigma_0}{6h}(1-x)^3 + \frac{\sigma_1}{6h}x^3, x \in [0,1],\\
S_2(x) = {} & \alpha_2(x-1) + \beta_2(2-x) + \frac{\sigma_1}{6h}(2-x)^3 + \frac{\sigma_2}{6h}(x-1)^3, x \in [1,2],\\
S_3(x) = {} & \alpha_3(x-x_2) + \beta_3(3-x) + \frac{\sigma_2}{6h}(3-x)^3 + \frac{\sigma_3}{6h}(x-2)^3, x \in [2,3].
\end{aligned}$$

A cubic spline example

We first solve for the σ_i using (3):

$$\frac{1}{6}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -18 \\ 0 \\ 0 \end{pmatrix}$$

$(2 - 2(1) + 0) = 0.$

This gives

$$\sigma_0 = 0, \quad \sigma_1 = -\frac{24}{5}, \quad \sigma_2 = \frac{6}{5}, \quad \sigma_3 = 0.$$

Now use (2) to get

$$\alpha_1 = \frac{14}{5}, \quad \alpha_2 = \frac{4}{5}, \quad \alpha_3 = 0.$$

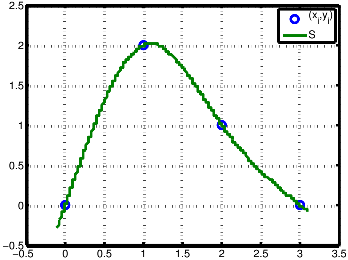
and

$$\beta_1 = 0 \quad \beta_2 = \frac{14}{5}, \quad \beta_3 = \frac{4}{5}.$$

The answer is

$$S(x) = \begin{cases} \frac{14}{5}x - \frac{4}{5}x^3, & x \in [0,1], \\ \frac{4}{5}(x-1) + \frac{14}{5}(2-x) - & \\ \qquad \frac{4}{5}(2-x)^3 + \frac{1}{5}(x-1)^3, & x \in [1,2], \\ \frac{4}{5}(3-x) + \frac{1}{5}(3-x)^3. & x \in [2,3]. \end{cases}$$

This is shown below.



 $f \in C^4[a,b] \Rightarrow f, f', f'', f'''$ & $f^{(4)}$ all continuous on $[a,b]$.

We state the following error estimates without proof. You don't need to know these, or be able to prove them. But you should be able to use them if required.

Theorem

If $f \in C^4[a,b]$ and S is its cubic spline interpolant on $N+1$ equally spaced points, then

Compare with linear splines (h^2).

$$\max_{x_0 \le x \le x_N} |f(x) - S(x)| = \|f - S\|_\infty \le \frac{5}{384} M_4 h^4,$$

$$\|f' - S'\|_\infty \le \frac{1}{24} M_4 h^3,$$

$$\|f'' - S''\|_\infty \le \frac{3}{8} M_4 h^2,$$

where $M_4 = \|f^{(iv)}\|_\infty$.

Example

Give an upper bound on the error for the cubic spline interpolant to $f = e^x$ on the interval $[-\mathbf{1}, \mathbf{1}]$ with $N = 10$ mesh points.

$f(x) = e^x$, so $f^{(iv)}(x) = e^x$, so $M_4 = e \cong 2.718$.

$h = \dfrac{x_N - x_0}{N} = \dfrac{1-(-1)}{10} = 0.2$.

So $\|f - S\|_\infty = \dfrac{5}{384}(2.718)(0.2)^4 \cong 5.6631 \times 10^{-5}$

(Compare with the error for linear spline: 1.4×10^{-2}).

on $[-1,1]$

Example

What is the smallest value of N that you must take to ensure that, if interpolating $f(x) = e^x$ on N equally sized intervals, the error is less that 10^{-8}? How does this compare with a linear spline interpolation (see corresponding example for linear splines).

We want $\left(\frac{5}{384}\right)\underbrace{(2.718)}_{M_4} h^4 \leq 10^{-8}$.

will find need $h \leq 0.02306$,

so $N \geq 86.7$. So take $N = 87$.

(Linear Spline: $N = 10{,}000$ approx !).

Recall the minimum energy property of linear splines. There is an analogous result for natural cubic splines.

Theorem

Let u be any function that interpolates f at ω^N, and is such that $u \in H^2(x_0, x_N)$. Then

$$\|S''\|_2 \leq \|u''\|_2.$$

The proof is not hard - it is analogous to the proof of Theorem 2.1.9.

Finished 13/2/17.