§3 Numerical Integration

§3.1 Introduction / Newton-Cotes / The Trapezium Rule

MA378/531 – Numerical Analysis II ("NA2")

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Problem

Given a real-valued function f that is continuous on $[a, b]$, can we find an estimate for

$$I(f) := \int_a^b f(x)dx?$$

And if we can, can we say how accurate it is?

Why bother?

- Many problems in applicable mathematics require definite integrals to be evaluated. (These methods were originally motivated by problems in astronomy).
- Evaluating them by finding the anti-derivative can be hard, and very hard to automate.
- Some times, although the function is integrable, its anti-derivative doesn't exist in a closed form.

The process of numerically estimating a definite integral is called ***Numerical Integration*** or ***Quadrature***.

The formulae we'll derive all look like

$$Q_n(f) := q_0 f(x_0) + q_1 f(x_1) + q_2 f(x_2) + \cdots + q_n f(x_n).$$

Here the points x_i are called *quadrature points* and the q_i are *quadrature weights*.

We need a way of choosing these.

The simplest approach is to take the points to be equally spaced, i.e., $x_i = a + hi$ where $h = (b-a)/n$.

How to choose the weights?

We've spent quite a while talking and thinking about approximating functions with polynomials. So why not find a polynomial interpolant to f and take the integral of that to be the answer? The appeal of this approach is due to the fact that

- Finding polynomial interpolants is easy.
- Integrating polynomials is easy.
- We can estimate the error easily (yet again, we'll make use of Cauchy's Theorem).

This leads to the **Newton-Cotes** methods, which are the subject of this section, and the next one. Later again, we'll look at more sophisticated methods, called ***Gaussian Methods*** which use non-uniformly spaced points.

Definition (Newton-Cotes quadrature)

The ***Newton-Cotes*** quadrature rule for $\int_a^b f(x)dx$ with $n+1$ points is derived by integrating exactly the polynomial of degree n that interpolates f at the $n+1$ equally spaced points $a = x_0 < x_1 < \cdots < x_n = b$. The method is written as

$$Q_n(f) := q_0 f_0 + q_1 f_1 + q_2 f_2 + \cdots + q_n f_n,$$

where we use the notation $f_k := f(x_k)$.

That is, the quadrature weights are chosen so that

$$Q_n(f) = \int_a^b p_n(x)dx,$$

where p_n is the polynomial of degree n that interpolates f at the $n+1$ quadrature points...

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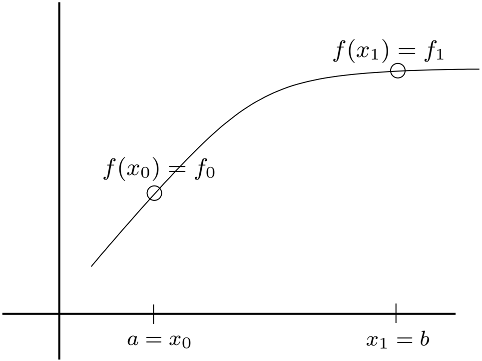
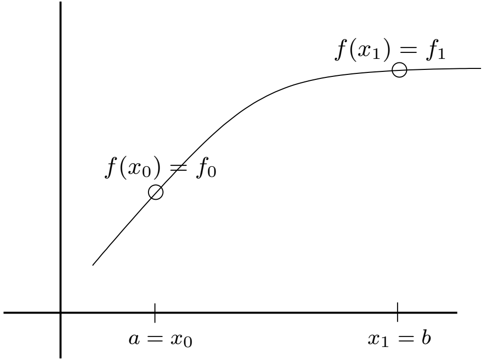
However, it turns out that we can compute the weights q_0, q_1, ..., q_n, ***without*** knowing p_n.

We'll do this for $n = 1$ in the next section, and $n = 2$ (the most interesting case) in Section 3.2.

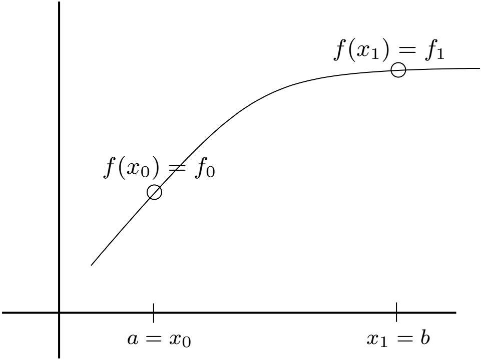
Suppose we wanted to estimate the integral of a function, f, shown below, on the interval $[a, b]$.



Method 1: We could try to estimate the area of the trapezium that fits under the graph:



Method 2: We could find p_1, the polynomial of degree 1 that interpolates f at $x = a$ and $x = b$:



Note that this shows that $q_i = \displaystyle\int_a^b L_i(x)dx$, where, as usual, the L_i are the Lagrange Polynomials.

Method 3: The third approach for generating the Trapezium Rule is called the *Method of Undetermined Coefficients*. Because the method is based on integrating a linear function we expect it to yield an exact solution for any constant or linear function (i.e., there should be no error). To keep the algebra simple, we'll take $a = 0$ and $b = 1$. So,

$$Q_1(f) = q_0 f(0) + q_1 f(1),$$

and, setting $f(x) \equiv 1$, and then $f(x) = x$ we get

Now we need to extend this to estimating $\int_a^b g(x)dx$ as follows:

Example

Use the trapezoid to estimate

$$\int_0^{\pi/4} \cos(x)dx.$$

Calculate the (exact) error $|\int_a^b f(x)dx - Q_1(f)|$.