Annotated notes from Monday, 19/2/2017

§3 Numerical Integration

§3.1 Introduction / Newton-Cotes / The Trapezium Rule

MA378/531 – Numerical Analysis II ("NA2")

February 2017

Problem

Given a real-valued function f that is continuous on $[a, b]$, can we find an estimate for

I is integral operator.

$$I(f) := \int_a^b f(x)dx?$$

ie, estimate the area under the curve, f, between a & b.

And if we can, can we say how accurate it is?

Why bother?

- Many problems in applicable mathematics require definite integrals to be evaluated. (These methods were originally motivated by problems in astronomy).
- Evaluating them by finding the anti-derivative can be hard, and very hard to automate.
- Some times, although the function is integrable, its anti-derivative doesn't exist in a closed form.

$Q(f) \approx I(f)$.

The process of numerically estimating a definite integral is called ***Numerical Integration*** or ***Quadrature***.

The formulae we'll derive all look like

$$Q_n(f) := q_0 f(x_0) + q_1 f(x_1) + q_2 f(x_2) + \cdots + q_n f(x_n).$$

Here the points x_i are called *quadrature points* and the q_i are *quadrature weights*.

We need a way of choosing these.

The simplest approach is to take the points to be equally spaced, i.e., $x_i = a + hi$ where $h = (b-a)/n$.

$x_0 = a, \quad x_1 = a + h, \quad x_2 = a + 2h \quad \cdots, \ldots, x_n = b.$

How to choose the weights?

We've spent quite a while talking and thinking about approximating functions with polynomials. So why not find a polynomial interpolant to f and take the integral of that to be the answer? The appeal of this approach is due to the fact that

- Finding polynomial interpolants is easy.
- Integrating polynomials is easy.
- We can estimate the error easily (yet again, we'll make use of Cauchy's Theorem).

This leads to the **Newton-Cotes** methods, which are the subject of this section, and the next one. Later again, we'll look at more sophisticated methods, called ***Gaussian Methods*** which use non-uniformly spaced points.

Definition (Newton-Cotes quadrature)

The ***Newton-Cotes*** quadrature rule for $\int_a^b f(x)dx$ with $n+1$ points is derived by integrating exactly the polynomial of degree n that interpolates f at the $n+1$ equally spaced points $a = x_0 < x_1 < \cdots < x_n = b$. The method is written as

$$Q_n(f) := q_0 f_0 + q_1 f_1 + q_2 f_2 + \cdots + q_n f_n, \quad = \sum_{i=0}^{n} q_i f_i$$

where we use the notation $f_k := f(x_k)$.

That is, the quadrature weights are chosen so that

$$Q_n(f) = \int_a^b p_n(x)dx,$$

where p_n is the polynomial of degree n that interpolates f at the $n+1$ quadrature points...

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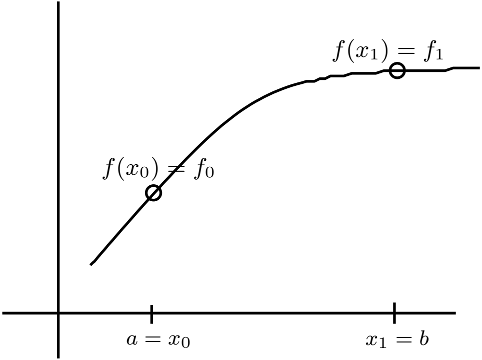
However, it turns out that we can compute the weights q_0, q_1, ..., q_n, ***without*** knowing p_n.

We'll do this for $n = 1$ in the next section, and $\underbrace{n = 2}$ (the most interesting case) in Section 3.2.

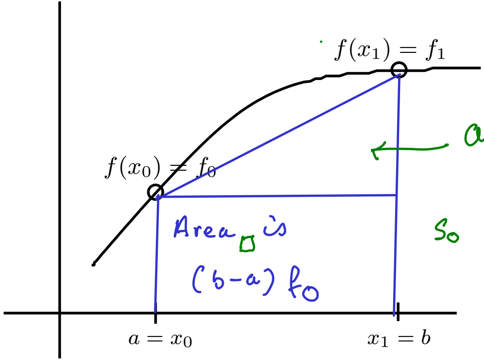
Tropezium Rule

Simpson's.

Suppose we wanted to estimate the integral of a function, f, shown below, on the interval $[a, b]$.



Method 1: We could try to estimate the area of the trapezium that fits under the graph:



Area$_\triangle$: $\frac{1}{2}(b-a)(f_1 - f_0)$

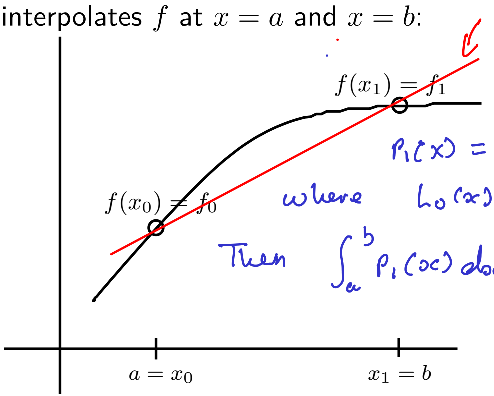
So Area$_\triangle$ + Area$_\square$ =

$$\frac{1}{2}(b-a)(f_1 - f_0) + (b-a)f_0$$

$$= \frac{1}{2}(b-a)(f_0 + f_1).$$

Geometrically intuitive, but hard to generalise & analyse.

Method 2: We could find p_1, the polynomial of degree 1 that interpolates f at $x = a$ and $x = b$:



(Lagrange form).

$p_1(x) = f_0 L_0(x) + f_1 L_1(x)$

where $L_0(x) = \dfrac{x-b}{a-b}$ $\quad L_1(x) = \dfrac{x-a}{b-a}$.

Then $\int_a^b p_1(x)\,dx = \int_a^b f_0 L_0(x) + f_1 L_1(x)\,dx$

$= f_0 \int_a^b L_0(x)\,dx + f_1 \int_a^b L_1(x)\,dx.$

Note that this shows that $q_i = \displaystyle\int_a^b L_i(x)dx$, where, as usual, the L_i are the Lagrange Polynomials.

Easy to generalise & analyse... but not so practical...

Method 3: The third approach for generating the Trapezium Rule is called the *Method of Undetermined Coefficients*. Because the method is based on integrating a linear function we expect it to yield an exact solution for any constant or linear function (i.e., there should be no error). To keep the algebra simple, we'll take $\boxed{a = 0 \text{ and } b = 1}$ So,

$$Q_1(f) = q_0 f(0) + q_1 f(1),$$

and, setting $f(x) \equiv 1$, and then $f(x) = x$ we get

$f(x)=1$: $Q_1(f) = q_0 + q_1$. Also $\int_0^1 f(x)\,dx = 1$ so $q_0 + q_1 = 1$

$f(x)=x$: $Q_1(f) = q_0(0) + q_1 = q_1$. Also $\int_0^1 x\,dx = \frac{1}{2}$. So

$q_1 = \frac{1}{2}$. Then $q_0 = \frac{1}{2}$ too.

Now we need to extend this to estimating $\int_a^b g(\cancel{x}\,t)\,d\cancel{x}\,t$ as follows:

Define the map from $[0,1]$ to $[a,b]$ as

$t = a + (b-a)x$. Note that $dt = (b-a)dx$

Now let $f(x) = g(\underbrace{a + (b-a)x}_{t})$.

So

$$\int_a^b g(t)\,dt = \int_0^1 f(x)\,\underbrace{(b-a)\,dx}_{dt}. = (b-a)\int_0^1 f(x)\,dx$$

Also $f(0) = g(a)$, $f(1) = g(b)$

So $\frac{1}{2}f(0) + \frac{1}{2}f(1)$. $\rightsquigarrow$ $\frac{(b-a)}{2}\left(g(a) - g(b)\right)$,

Example

Use the trapezoid to estimate

$$\int_0^{\pi/4} \cos(x)dx.$$

Calculate the (exact) error $|\int_a^b f(x)dx - Q_1(f)|$.

$f(x) = \cos(x) \quad a = 0, \quad b = \pi/4.$

$I(f) = \sin(x)\Big|_0^{\pi/4} = \frac{1}{\sqrt{2}} \simeq 0.7071.$

Then $Q(f) = \frac{b-a}{2}(f(a) - f(b)) = \frac{(\pi/4)}{2}(\cos(0) + \cos(\pi/4))$

$= 0.67038$

Error is $\boxed{0.0367}$