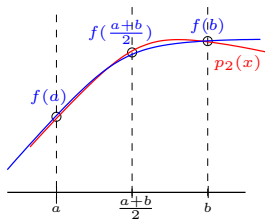


§3 Numerical Integration

§3.2 Simpson's Rule

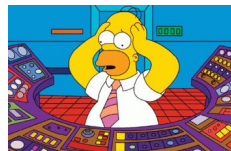
MA378/531 – Numerical Analysis II (“NA2”)

February 2017



T. Simpson's Rule:

$$\int_a^b f(x) dx \approx \frac{b-a}{4} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right)$$



H. Simpson's Rule:

"If something is hard to do, it is not worth doing".

Following on from the Trapezium Rule, we'll consider the **3-point Newton-Cotes scheme**¹. which is based on integrating the quadratic interpolant to $f(x)$.

Simpson's Rule

$$Q_2(f) = \frac{b-a}{6} (f(a) + 4f(\frac{a+b}{2}) + f(b)). \quad (1)$$

¹Thomas Simpson, 1710–1761. One of the most distinguished of a group of itinerant lecturers who taught in the London coffee-houses, Hutton (famous text-book writer) said of him

It has been said that Mr Simpson frequented low company, with whom he used to guzzle porter and gin: but it must be observed that the misconduct of his family put it out of his power to keep the company of gentlemen, as well as to procure better liquor.

The method was known well before Simpson's time: it had been used by Cavalieri (a student of Galileo) in 1639, James Gregory, Johannes Kepler, and others.

To show how to derive it, we'll use the ***Method of Undetermined Coefficients*** again.

First restrict our attention to approximating $\int_0^1 g(x)dx$. This method should be exact for all constant, linear and quadratic polynomials. Taking

$$g(x) \equiv 1, \quad g(x) = x \quad \text{and} \quad g(x) = x^2,$$

we get the set of equations:

This is easily solved giving

$$\int_0^1 g(x)dx \approx \frac{1}{6}g(0) + \frac{2}{3}g(1/2) + \frac{1}{6}g(1). \quad (2)$$

To extend this to the interval $[a, b]$, we again use a change of variables to get the general Simpson's Rule (1) .

Example

Use Simpson's rule to estimate $\int_0^{\pi/4} \cos(x) dx$, and calculate the (exact) error $|\int_a^b f(x) dx - Q_2(f)|$.

We'll now derive error estimates for general Newton-Cotes methods, and look at the specific cases of the Trapezium and Simpson's rules.

Theorem

Let $M_{n+1} := \max_{a \leq x \leq b} |f^{(n+1)}(x)|$, and $\pi_{n+1}(x)$ be the usual nodal polynomial. Define

$$\mathcal{E}_n := \left| \int_a^b f(x) dx - Q_n(f) \right|.$$

Then

$$\mathcal{E}_n \leq \frac{M_{n+1}}{(n+1)!} \int_a^b |\pi_{n+1}(x)| dx,$$

The proof just comes directly Cauchy's Theorem.

Theorem

For the **Trapezium Rule**, Q_1 ,

$$\mathcal{E}_1 \leq \frac{(b-a)^3}{12} M_2. \quad (3)$$

The proof is an exercise.

Example

Use (3) to get an upper bound on the error for the estimate of $\int_0^{\pi/4} \cos(x) dx$ using the Trapezium rule. How does this compare with the actual error?

Example

If use the Trapezium Rule to estimate the integral of x^2 on the interval $[0, 1]$ we get

$$\int_0^1 x^2 dx = \frac{1}{3} \quad \text{and} \quad Q_1(x^2) = \frac{1}{2}(0 + 1) = \frac{1}{2}.$$

So the error is $1/6$, exactly as the theory predicts.

One can also use our theorem to show that for Simpson's Rule

$$\mathcal{E}_2 \leq \frac{(b-a)^4}{196} M_3, \quad (4)$$

but don't bother because, although correct, it is not *sharp* (that is, it is pessimistic).

Example

Use (4) to get an upper bound on the error for the estimate of $\int_0^{\pi/4} \cos(x) dx$ using **Simpson's** rule. How does this compare with the actual error?

Example

We expect Simpson's Rule to give *exactly* the right answer for integrals of constant, linear and quadratic functions. If we take $f(x) = x^3$, $a = 0$ and $b = 1$, then formula above suggests that (approx) $\mathcal{E}_2 \leq 0.03$. But ...