§3 Numerical Integration

§3.3 Precision and Composition

MA378 – Numerical Analysis II (“NA2”)

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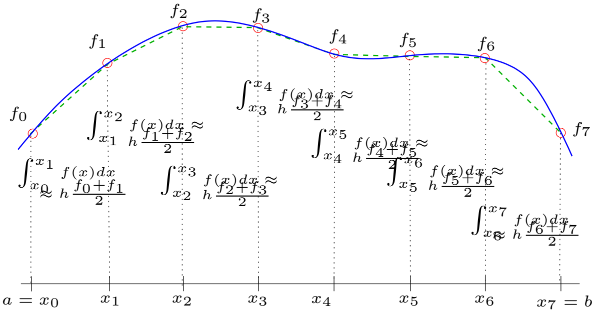

We know that Simpson's Rule applied to approximating $\int_0^1 x^3 dx$ yields exactly the right answer (i.e., the error is zero).

We now claim that Simpson's Rule is exact for ***any*** polynomial of degree 3 or less.

Details: In the online notes, it is shown that, if f is any cubic polynomial, then, for any interval $[a, b]$,

$$\int_a^b f(x)dx = Q_2(f).$$

Please read that, and make sure you understand it. In class we will take a simpler, but less direct approach, by setting $a = -1$ and $b = 1$, and showing that, if f is any cubic polynomial, then $\int_{-1}^1 f(x)dx = Q_2(f)$.

When approximating of $\int_{-1}^{1} f(x)dx$ with Simpson's Rule, the formula is

$$Q_2(f) = \frac{2}{6}\Big(f(-1) + 4f(0) + f(1)\Big).$$

Since the method can be derived by integrating the quadratic that interpolates $f(x)$ at the three points -1, 0, and 1, it is clearly exact for all quadratics. This fact can also be derived from the error estimate:

$$\mathcal{E}_2 \le \frac{(b-a)^4}{196} M_3.$$

Precision

In Section 3.2, we discussed a nïeve attempt to derive an upper bound for the error in Simpson's rule leading to

$$\mathcal{E}_2 \leq \frac{(b-a)^4}{196} M_3.$$

This is not wrong – just not sharp. For example it does not give that Simpson's Rule is exact for all cubic. The sharp result is

Theorem

$$|E_2(x)| = |\int_a^b f(x)dx - Q_2(f(x))| \leq \frac{(b-a)^5}{2880} M_4.$$

For the proof see the text book (Theorem 7.2 of Suli and Mayers). Instead of working through it in class we'll prove a more general version of a consequence it.

Precision

Definition (Precision of a Quadrature Rule)

A quadrature rule has **precision** n if it is exact for all polynomials of degree n or less. That is, the rule $Q(f)$ has precision n if

$$Q(p_n) = \int_a^b p_n(x)dx \quad \text{for all } p_n \in \mathcal{P}_n.$$

Example

By construction, the $(n+1)$-point Newton-Cotes rule has precision n.

Theorem

If $Q_{2k}(\cdot)$ is a Newton-Cotes quadrature rule on $2k+1$ points, then $Q_{2k}(\cdot)$ has in fact precision $2k+1$.

Proof: Let p_{n+1} be a polynomial of degree $n+1$. We wish to show that $Q_n(p_{n+1}) = \int_a^b p_{n+1}(x)dx$. We can take $a=-1$, $b=1$ because a simple linear transformation can be used to map to an arbitrary interval.

Also, since the quadrature points are equally spaced on $[-1,1]$ we have that $x_i = -x_{n-i}$.

Furthermore (see exercises) the quadrature weights are symmetric: $q_i = q_{n-i}$.

Suppose that we want to estimate $\int_a^b f(x)dx$ and the Trapezium rule is not sufficiently accurate. We could try Simpson's Rule, which should be better. Failing that, we could try a 4-point rule, based on integrating the p_4 interpolant, or a five-point rule, based on integrating p_5.

However, quite apart from the fact that it might be tedious to derive these rules, we know (Runge's example again!) that high-order polynomial interpolation can be very inaccurate.

It is better to use a ***Composite Rule***. This is analogous to the idea behind piecewise linear.

For the **Composite Trapezium Rule** we divide $[a, b]$ into N intervals of size $h = (b - a)/N$. Applying the Trapezium Rule on each interval $[x_{i-1}, x_i]$ we get

$$\int_{x_{i-1}}^{x_i} f(x)dx \approx h\frac{f_{i-1} + f_i}{2}.$$

Summing for the n intervals we get

$$\int_a^b f(x)dx \approx \frac{b-a}{N}\left(\frac{f_0}{2} + f_1 + f_2 + \cdots + f_{N-1} + \frac{f_N}{2}\right). \qquad (1)$$

Composite Rules