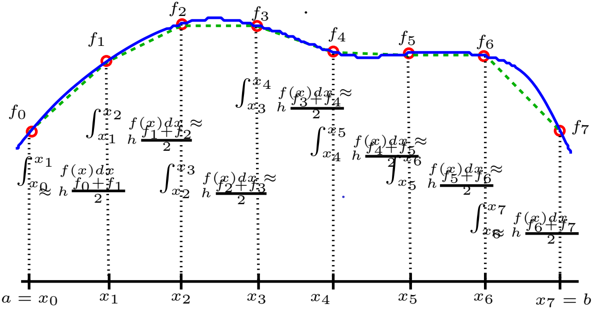
Started & finished 27/2/17.

§3 Numerical Integration

§3.3 Precision and Composition

MA378 – Numerical Analysis II ("NA2")

February 2017



We know that Simpson's Rule applied to approximating $\int_0^1 x^3 dx$ yields exactly the right answer (i.e., the error is zero).

We now claim that Simpson's Rule is exact for ***any*** polynomial of degree 3 or less.

Details: In the online notes, it is shown that, if f is any cubic polynomial, then, for any interval $[a, b]$,

$$\int_a^b f(x)dx = Q_2(f).$$

Please read that, and make sure you understand it. In class we will take a simpler, but less direct approach, by setting $a = -1$ and $b = 1$, and showing that, if f is any cubic polynomial, then $\int_{-1}^{1} f(x)dx = Q_2(f)$.

When approximating of $\int_{-1}^{1} f(x)dx$ with Simpson's Rule, the formula is

$$Q_2(f) = \frac{2}{6}\Big(f(-1) + 4f(0) + f(1)\Big).$$

ie $a = -1$, $b = +1$.

Since the method can be derived by integrating the quadratic that interpolates $f(x)$ at the three points -1, 0, and 1, it is clearly exact for all quadratics. This fact can also be derived from the error estimate:

$$\mathcal{E}_2 \leq \frac{(b-a)^4}{196} M_3.,$$

from Cauchy's Thm

where $M_3 = \max_{a \leq x \leq b} |f'''(x)|$.

But if f is quadratic, then f' is liner, f'' is constant & $f'''(x) \equiv 0$.

Let f be an arbitrary cubic:

$$f(x) = \underbrace{c_0 + c_1 x + c_2 x^2}_{q_2(x)} + c_3 x^3 \,, \quad \text{and write}$$

it as

$$f(x) = q_2(x) + c_3 x^3.$$

Then
$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} q_2(x)\,dx + \int_{-1}^{1} c_3 x^3\,dx$$
$$= \int_{-1}^{1} q_2(x)\,dx + \frac{1}{4} c_3 \, x^4 \Big|_{-1}^{+1}$$
$$= \int_{-1}^{1} q_2(x)\,dx + \frac{1}{4} c_3 \big(\underbrace{1^4 - (-1)^4}_{0}\big)$$
$$= \int_{-1}^{1} q_2(x)\,dx$$

See board.

(Briefly) Also

$$Q_2(f) = Q_2(q_2) + c_3 \, Q_2(x^3).$$

$$= \underbrace{\int_{-1}^{1} q_2(x)\,dx}_{\text{because } Q_2 \text{ is exact for Quadratics}} + c_3 \frac{2}{6} \underbrace{\left((-1)^3 + 4(0)^3 + (1)^3\right)}_{= -1 + 1 = 0.}$$

So $Q_2(f) = \int_{-1}^{1} q_2(x)\,dx = \int_{-1}^{1} f(x)\,dx$. So Q_2 is exact for cubics on $[-1,1]$.

In Section 3.2, we discussed a nïeve attempt to derive an upper bound for the error in Simpson's rule leading to

$$\mathcal{E}_2 \leq \frac{(b-a)^4}{196} M_3.$$

more of 4th deriv.

This is not wrong – just not sharp. For example it does not give that Simpson's Rule is exact for all cubic. The sharp result is

Theorem

$$|E_2(x)| = |\int_a^b f(x)dx - Q_2(f(x))| \leq \frac{(b-a)^5}{2880} M_4.$$

For the proof see the text book (Theorem 7.2 of Suli and Mayers). Instead of working through it in class we'll prove a more general version of a consequence it.

Definition (Precision of a Quadrature Rule)

A quadrature rule has **precision** n if it is exact for all polynomials of degree n or less. That is, the rule $Q(f)$ has precision n if

$$Q(p_n) = \int_a^b p_n(x)dx \quad \text{for all } p_n \in \mathcal{P}_n.$$

Example

By construction, the $(n+1)$-point Newton-Cotes rule has precision n.

This is because it is derived by exactly integrating the poly of degree n that interpolates f at n+1 equally spaced points. If f is a poly of degree n, it is its own interpolant.

Theorem

If $Q_{2k}(\cdot)$ is a Newton-Cotes quadrature rule on $2k+1$ points, then $Q_{2k}(\cdot)$ has in fact precision $2k+1$. ie an odd number of points.

Proof: Let p_{n+1} be a polynomial of degree $n+1$. We wish to show that $Q_n(p_{n+1}) = \int_a^b p_{n+1}(x)dx$. We can take $a=-1$, $b=1$ because a simple linear transformation can be used to map to an arbitrary interval.

Also, since the quadrature points are equally spaced on $[-1,1]$ we have that $x_i = -x_{n-i}$.

Furthermore (see exercises) the quadrature weights are symmetric: $q_i = q_{n-i}$. (Hint: compare $L_0(x)$ with $L_n(x)$ where L_0 & L_n are Lagrange Polys).

(on board we saw that ...)

$$\therefore \int_{-1}^{1} p_{n+1}(x)\,dx = \int_{-1}^{1} p_n(x)\,dx + \underbrace{\frac{c_{n+1}}{n+2}\left((-1)^{n+2} - (1)^{n+2}\right)}_{0}$$

Similarly,

$$Q_n(p_{n+1}) = Q_n(p_n) + Q_n\left(c_{n+1}\, x^{n+1}\right)$$

$$= \int_{-1}^{1} p_n(x)\,dx = \int_{-1}^{1} p_{n+1}(x)\,dx,$$

because

$$Q_n(x^{n+1}) = \sum_{i=0}^{n} q_i\, x_i^{n+1} = \sum_{i=0}^{n/2-1} q_i\, x_i^{n+1}$$

$$+ \sum_{i=0}^{n/2-1} q_{n-i}\, x_{n-i}^{n+1} + q_{n/2}\,(0)^{n+1} \ldots$$

Suppose that we want to estimate $\int_a^b f(x)dx$ and the Trapezium rule is not sufficiently accurate. We could try Simpson's Rule, which should be better. Failing that, we could try a 4-point rule, based on integrating the p_4 interpolant, or a five-point rule, based on integrating p_5.

However, quite apart from the fact that it might be tedious to derive these rules, we know (Runge's example again!) that high-order polynomial interpolation can be very inaccurate.

It is better to use a ***Composite Rule***. This is analogous to the idea behind piecewise linear.

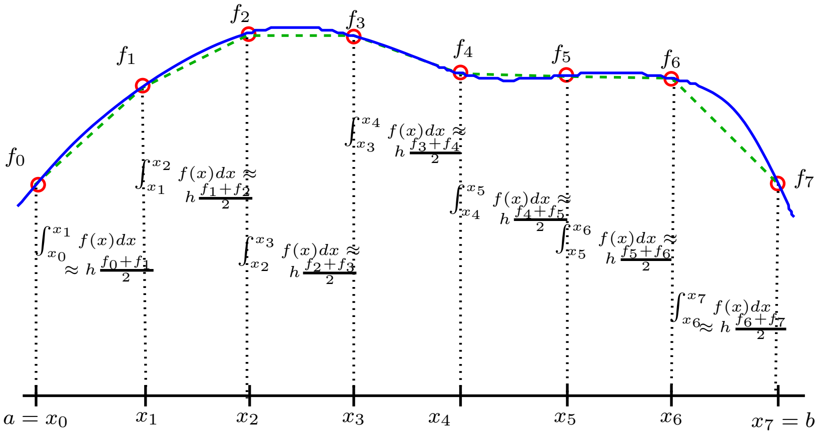
For the **Composite Trapezium Rule** we divide $[a,b]$ into N intervals of size $h=(b-a)/N$. Applying the Trapezium Rule on each interval $[x_{i-1},x_i]$ we get

$$\int_{x_{i-1}}^{x_i} f(x)dx \approx h\frac{f_{i-1}+f_i}{2}.$$

Summing for the n intervals we get

$$\int_a^b f(x)dx \approx \overbrace{\frac{b-a}{N}}^{h}\left(\frac{f_0}{2}+f_1+f_2+\cdots+f_{N-1}+\frac{f_N}{2}\right). \qquad (1)$$

$$\frac{h}{2}(f_0+f_1) + \frac{h}{2}(f_1+f_2) + \frac{h}{2}(f_2+f_3) + \cdots + \frac{h}{2}(f_{n-1}+f_N)$$



Note $\int_a^b f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx$

$+ \dots + \int_{x_{N-1}}^{x_N} f(x)\,dx$