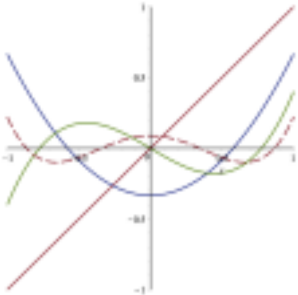
§3 Numerical Integration

§3.5 Orthogonal Polynomials

MA378 – Numerical Analysis II ("NA2")

March 2017



Started Monday, 6 March 2017
No Lecture on Wed, 8 March (Class Test)
Finished Wednesday, 15 March.

High order Newton-Cotes methods are of little use because of the problems associated with interpolation be high degree polynomials at equally spaced points. However, high-order Gaussian methods are very useful.

Driving such methods by undetermined coefficients is not practical, however. There is a simpler way, but some mathematical preliminaries are required, including the ideas of **vector spaces** and **inner products**.

Also: the set of all polynomials of degree n is a V.S.

Definition (Vector Space)

V is a *vector space* (a.k.a., a *linear space*) over a field F (e.g, the real or complex numbers) if for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and $a, b \in F$:

- (i) $\boldsymbol{u} + \boldsymbol{v} \in V$ (closed under addition)
- (ii) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ (Commutativity)
- (iii) $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$ (Associativity)
- (iv) V has a zero vector $\mathbf{0}$ such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$.
- (v) $-\boldsymbol{u} \in V$
- (vi) $a\boldsymbol{u} \in V$
- (vii) $a(b\boldsymbol{u}) = (ab)\boldsymbol{u}$
- (viii) F contains 0 and 1 such that $1\boldsymbol{u} = \boldsymbol{u}$, $0\boldsymbol{u} = \mathbf{0}$.
- (ix) $a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$, and $(a + b)\boldsymbol{u} = a\boldsymbol{u} + b\boldsymbol{u}$.

Examples: $\mathbb{R}$, $\mathbb{R}^2$ .. , $\mathbb{R}^n$ are all vector spaces, but $\mathbb{R}^2$ & $\mathbb{R}^3$ are the most familiar.

Definition (Inner Product)

Let V is a real vector space. An **Inner Product** (IP) is a real-valued function $(\cdot,\cdot)$ on $V \times V$ such that, for all $f, g, h \in V$,

(i) $(f+g,h) = (f,h) + (g,h)$,

(ii) $(\lambda f, g) = \lambda(f,g)$, for $\lambda \in \mathbb{R}$.

(iii) $(f,g) = (g,f)$,

(iv) $(f,f) \geq 0$. $\quad (f,f) = 0 \Leftrightarrow f \equiv 0$.

Example

Let $\mathbb{R}^n$ be our vector space, with $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^T$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n)^T$. Then the following is an inner product:

Eg, $n = 3$,

$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad y = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$$(\boldsymbol{x}, \boldsymbol{y}) := \sum_{i=1}^{n} x_i y_i, \;=\; x^T y$$

$x^T y = (1)(1) + (2)(-2) + (3)(1) = 0$

$y^T y = 1^2 + (-2)^2 + 1^2 = 6,$

Example

The set of real-valued functions that are continuous and defined on the ionterval $[a,b]$, denoted $C[a,b]$, is a vector space. And

$$(f,g) := \int_a^b f(x)g(x)dx, \tag{1}$$

is an inner product.

Check properties (i)–(iv).

here: $f = f(x)$
$g = g(x)$, etc.

$$(f+g,h) = \int_a^b (f+g)h\,dx$$

$$= \int_a^b fh + gh\,dx = \int_a^b fh\,dx + \int_a^b gh\,dx$$

$$= (f,h) + (g,h)$$

Check properties (ii)–(iv) yourself.

(See Lecture 23 of Stewart's "Afternotes" for more details).

Definition (Monic Polynomial)

A polynomial is *monic* if the coefficient of its leading term is 1.

Examples: These are all monic:

$1, \quad x-2, \quad x^3 - 4x^2 - 1000,$

$3x^3 + 4x^2 + x^6$

These are not:

$2, \quad 2x-1, \quad x^3 - 4x^2 \underbrace{- 1000x^5}_{} + 1$

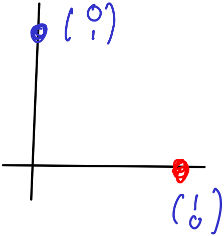
These are: $1, x, x^2, x^3, x^4,$ etc.

Definition

Two elements a, b, of a vector space are *orthogonal* with respect to a given inner product $(\cdot, \cdot)$ if $(a, b) = 0$. $a \perp b$

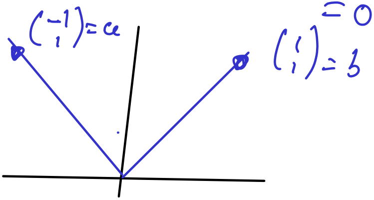
Example: In $\mathbb{R}^2$, $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are orthogonal

$$(a, b) = (1)(0) + (0)(1) = 0.$$



Eg $a = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Then $(a, b) = (-1)(1) + (1)(1) = 0$



Eg In $\mathbb{R}^3$, $a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

write "$a \perp b$"

Example

Take the space of polynomials of degree 2 or less and the IP

$\mathbb{P}_2$.

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx.$$

Let $p(x) \equiv 1$, $q(x) \equiv x$, $r(x) \equiv x^2 - 1/3$, and $f(x) = 3x - 4$

Details: $(p, q) = \int_{-1}^{1} (1)(x)\,dx = \frac{1}{2}x^2 \Big|_{-1}^{1} = \frac{1}{2}(1 - (-1)^2) = 0.$

So $p \perp q$.

$(p, r) = \int_{-1}^{1} (1)(x^2 - \frac{1}{3})\,dx = \frac{1}{3}x^3 - \frac{1}{3}x \Big|_{-1}^{1} = \frac{1}{3} - \frac{1}{3} - \frac{1}{3} + \frac{1}{3} = 0.$

So $p \perp r$. $(q, r) = \int_{-1}^{1} x^3 - \frac{1}{3}x\,dx = \frac{1}{4}x^4 - \frac{1}{6}x^2 \Big|_{-1}^{1}$

$= \frac{1}{4} - \frac{1}{6} - \frac{1}{4} + \frac{1}{6} = 0.$ So $q \perp r$.

(see board).

Sequence of Orthogonal Monic Polynomials

As given above, a polynomial is ***monic*** if the coefficient of the leading term is 1:

$$p_n = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-1} + \cdots + c_1 x + c_0.$$

We'll now look at a sequence of such polynomials

$$\{\widetilde{p}_0, \widetilde{p}_1, \widetilde{p}_2, \dots, \widetilde{p}_n, \dots\}$$

that have the property they are orthogonal to each other:

$$(\widetilde{p}_i, \widetilde{p}_j) := \int_a^b \widetilde{p}_i(x)\widetilde{p}_j(x)dx = 0 \qquad \text{if } i \neq j.$$

We want to establish some important facts about monic polys:

- A set of monic polys of degrees $1, \dots, n$, forms a basis for $\mathcal{P}_n$.
- If the members of that set are orthogonal to each other, then they are orthogonal to *all* polynomials of lower degree.
- We can construct such as set.

Finished here

6/3/17.

Lemma

Let $\{p_i\}_{i=0}^n$ be a sequence of polynomials where each p_i is monic an exactly of degree i. This sequence forms as basis for $\mathcal{P}_n$.

Proof: Usually we write a polynomial of degree n as

$$q(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

we want to show we can write it as

$$q(x) = b_n p_n + b_{n-1} p_{n-1} + \cdots + b_1 p_1 + b_0 p_0 \quad p_0 = 1$$

where the b_i are unique. If $n=0$, then $b_0 = a_0$. Now assume true for $n-1$. If q has degree n, then $b_n = a_n$ since p_n is monic. Also

$q - b_n p_n$ has degree $n-1$. So, by induction, $b_{n-1}, b_{n-2}, \ldots, b_0$ exist & are unique.

Sequence of Orthogonal Monic Polynomials (11/16)

This lemma means that if q is a polynomial of degree n then it can be written uniquely as a linear combination of the $\tilde{p}_i$:

$$q(x) = \sum_{i=0}^{n} a_i \tilde{p}_i(x),$$

for some unique choice of the real coefficients a_i.

Definition

The sequence $\{\widetilde{p}_i\}_{i=0}^{n}$ is a sequence of *monic, orthogonal* polynomials if each $\widetilde{p}_i$ is monic and *exactly* of degree i and

$$(\widetilde{p}_i, \widetilde{p}_j) = 0 \quad \text{if } i \neq j.$$

Lemma

If $\widetilde{p}_j \in \{\tilde{p}_i\}_{i=0}^{\infty}$ then $\widetilde{p}_j$ is orthogonal to all polynomials of degree less than j.

Proof: Suppose that q is a polynomial of degree less than j. Then we can write q as

$$q(x) = \sum_{i=0}^{j-1} a_i \hat{p}_i(x).$$

Then $(\hat{p}_j, q) = \left(\hat{p}_j, \sum_{i=0}^{j-1} a_i \hat{p}_i\right)$

$$= \sum_{i=0}^{j-1} a_i (\hat{p}_j, \hat{p}_i) = 0,$$

Theorem

The sequence $\{\tilde{p}_i\}_{i=0}^{\infty}$ exists and can be constructed as follows: Let α and β be defined as

$$\alpha_{n+1} = \frac{(x\widetilde{p}_n, \widetilde{p}_n)}{(\widetilde{p}_n, \widetilde{p}_n)}, \quad \text{and} \quad \beta_{n+1} = \frac{(x\widetilde{p}_n, \widetilde{p}_{n-1})}{(\widetilde{p}_{n-1}, \widetilde{p}_{n-1})},$$

then the sequence is given by

$$\alpha_1 = \frac{(x\hat{p}_0, \hat{p}_0)}{(\hat{p}_0, \hat{p}_0)}$$

$$\widetilde{p}_0(x) \equiv 1, \qquad \widetilde{p}_1(x) = x - \alpha_1$$

and

$$\widetilde{p}_{n+1}(x) = (x - \alpha_{n+1})\widetilde{p}_n(x) - \beta_{n+1}\widetilde{p}_{n-1}(x),$$

for $n \geq 1$.

The proof uses *Gram-Schmidt Orthogonalization*.

Example

If we use the inner product $(f, g) := \int_0^1 f(x)g(x)$ then the first 4 polynomials in the sequence are:

See tomorrow's lab!

Example

The zeros of $\widetilde{p}_2$ are ...

One of the ways of constructing Gaussian Quadrature rule $G_n(\cdot)$ on $n+1$ is to take the quadrature points as the roots of $\widetilde{p}_{n+1}$. We know (from the fundamental theorem of algebra) a polynomial of degree $n+1$ has exactly $n+1$ roots in $\mathbb{C}$ up to multiplicity.

However, the polynomials $\widetilde{p}$ have the special properties, established in the following lemma. (*A slightly different proof of these facts is given in Thm 9.4 of Suli and Mayers.*).

Not every polynomial of degree $n+1$ has $n+1$ real zeros (let alone in $[a,b]$)

Eg $p_2 = x^2$ has just 1 zero

$p_2 = x^2+1$ has no real zeros.

Lemma

Let $\widetilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty} = \{\widetilde{p}_0, \widetilde{p}_1, \dots,\}$ be the set of monic polynomials that are orthogonal with respect to the (usual) inner product.

(i) *The zeros of each $\widetilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty}$ are simple (not repeated).*

(ii) *All the zeros of $\widetilde{p}_i$ are real numbers in the interval $[a, b]$.*

Proof of (i). Suppose $\hat{p}_i$ had a repeated root at x^*. Then we could write

$\hat{p}_i(x) = (x - x^*)^2 q(x)$ where q has degree $i-2$. But $0 = (\hat{p}_i, q)$ because $\deg(q) < \deg(\hat{p}_i)$

So $0 = \int_a^b (x - x^*)^2 q(x) q(x)\, dx > 0$. Contradiction!

Lemma

Let $\widetilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty} = \{\widetilde{p}_0, \widetilde{p}_1, \ldots,\}$ be the set of monic polynomials that are orthogonal with respect to the (usual) inner product.

(i) *The zeros of each $\widetilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty}$ are simple (not repeated).*

(ii) *All the zeros of $\widetilde{p}_i$ are real numbers in the interval $[a, b]$.*

(ii) Suppose that $\tilde{p}_i$ has only $k<i$ zeros in $[a,b]$. Then, there is a polynomial q such that $\tilde{p}_i(x) = q(x)\,r(x)$ where r has degree $k<i$ & q does not change sign on $[a,b]$. Since degree(r) < degree$(\tilde{p}_i)$,

$$0 = (\tilde{p}_i, r) = (qr, r) = \int_a^b q(x)\,(r(x))^2\,dx \neq 0$$

because q does not change sign on $[a,b]$ & $(r(x))^2 \geq 0$,