§3 Numerical Integration

# §3.7 Gaussian Quadrature via Orthogonal Polynomials

MA378/531 – Numerical Analysis II ("NA2")

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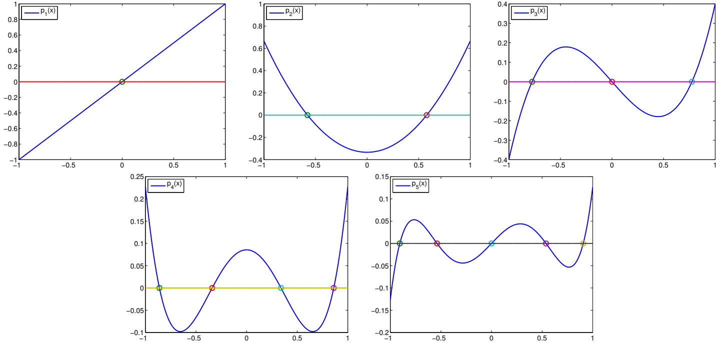


# Gaussian Quadrature via Orthogonal Polys 

At the start of this section, we introduced the using Gaussian Quadrature technique for estimating integrals

$$\int_a^b f(x)dx \approx G_n(f) := \sum_{k=0}^{n} w_k f(x_k),$$

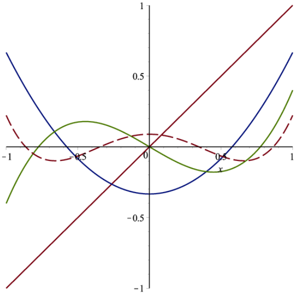
where the points $x_k$ and weights $w_k$ are chosen to maximise the precision of the method.

We now have an equivalent way of defining the method, $G_n(\cdot)$, and deriving the coefficients...

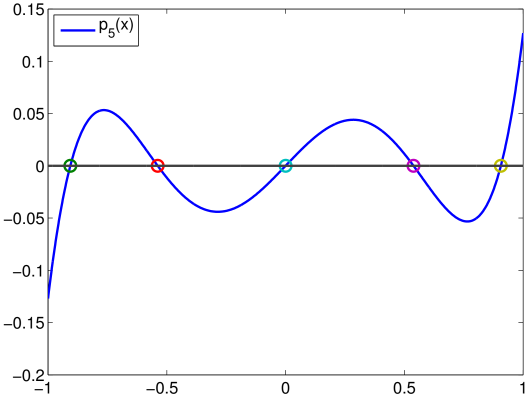
(a) Write down the set of monic polynomials $\{\widetilde{p}_0, \widetilde{p}_1, \ldots, \widetilde{p}_{n+1}\}$ that is orthogonal with respect to the inner product

$$(u, v) := \int_a^b u(x)v(x)dx.$$

Note that $a$ and $b$, the limits of integration for the IP, are the same as for the integral we are trying to estimate.



(b) We know that $\widetilde{p}_{n+1}$ has $n+1$ zeros in the interval $[a, b]$. Let these be the quadrature points of the method.
For example, the polynomial $\widetilde{p}_5 = x^5 - (10/9)x^3 + (5/21)x$ is shown below, with its zeros highlighted.



(c) Take the quadrature weights to be $w_k = \int_a^b L_k(x)dx$, where the $L_k$ are the usual Lagrange polynomials for this set of points.

The key property of this method in stated in the following theorem.

**Theorem**

*Let $x_0, \ldots, x_n$ to be the zeros of $\widetilde{p}_{n+1}$, the $(n+1)$th polynomial in the sequence of orthogonal monic polynomials $\{\tilde{p}_i\}_{i=0}^{\infty}$. Set*

$$G_n(f) = w_0 f(x_0) + \cdots + w_n f(x_n) \qquad \text{where } w_i = \int_a^b L_i(x)dx. \tag{1}$$

*Then $G_n(f)$ has precision $2n+1$.*

The proof of this is an exercise.

Our final task associated with numerical integration is to prove that, as $n \to \infty$, so $G_n(f) \to \int_a^b f(x)dx$. We won't do this in full detail, but the key ideas will be presented.

We would like to prove the following error estimate, which is closely related to Theorem 1.5.2 (error in Hermitian interpolation):

**Theorem**

$$\int_a^b f(x)dx - G_n(f) = \frac{f^{(2n+2)}(\tau)}{(2n+2)!} \int_a^b \pi_{n+1}(x)^2 dx.$$

***Idea:*** show that Gaussian Quadrature is the same as integrating the Hermite interpolant to $f$, even though $f'(x_k)$ is not involved.

***Moreover, want to show:*** $G_n(f) \to \displaystyle\int_a^b f(x)dx$ as $n \to \infty$.

To do this we need to establish several facts. In each case we make use of 1, a consequence of which is that, if $f$ is a polynomial of degree at most $2n-1$, then

$$\int_a^b f(x)dx = \sum_{k=0}^{n} w_k f(x_k).$$

Recall the basis functions that we use for ***Hermite Interpolation***:

$$H_i(x) = [L_i(x)]^2(1 - 2L_i'(x_i)(x - x_i)),$$

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

Now we can deduce the error estimate:

Next we want to show that each of the $w_k$ are positive. From (1) we have that the Gaussian Quadrature weights are $w_k = \int_a^b L_k(x)dx$ where the $L_k$ are the usual Lagrange Polynomials:

$$L_k(x_j) = \begin{cases} 1 & k = j \\ 0 & k \neq j, \end{cases}$$

associated with the Gaussian interpolation points. Then in fact....

So, since

$$w_i = \int_a^b [L_i(x)]^2 dx,$$

we have that all the $w_k$ are all positive. It follows directly that $0 < w_k < (b - a)$, for $k = 0, 1, 2, \ldots, n$:

Section 10.4 of Süli and Mayers also covers this, though from a different angle. One of the most interesting aspects of this theory is given in Theorem 10.2 of that book:

**Theorem**

$$\lim_{n\to\infty} G_n(f) = \int_a^b f(x)dx.$$

(*Here we'll just given an out line of the proof. You are not required to know this for the exam*).

The ***Weierstrass approximation theorem***, tells us that, for any $\epsilon > 0$, there exists a polynomial $p$ such that $|f(x) - p(x)| \leq \epsilon$. Let $n$ be the degree of this polynomial. Let $G_n(\cdot)$ be the $n+1$ point Gaussian Quadrature rule. Then

$$\int_a^b f(x)dx - G_n(f) = \int_a^b f(x) - p(x)dx + \int_a^b p(x)dx - G_n(p) + G_n(p) - G_n(f).$$

But because $G_n(\cdot)$ is exact for polynomials of degree $n$, $\int_a^b p(x)dx - G_n(p) = 0$. Using this, and the triangle inequality,

$$\left| \int_a^b f(x)dx - G_n(f) \right| \leq \left| \int_a^b f(x) - p(x)dx \right| + \left| G_n(p) - G_n(f) \right|.$$

But

$$\left| \int_a^b f(x) - p(x) dx \right| \leq \int_a^b \varepsilon dx = \varepsilon(b-a).$$

Also,

$$|G_n(f) - G_n(p)| = |G_n(p-f)| = \\ |\sum_{k=0}^{n} w_k(f(x_k) - p(x_k))| = \\ \sum_{k=0}^{n} w_k |f(x_k) - p(x_k)|,$$

because all the $w_i$ are positive. Then, using that $\sum_k w_k = b - a$ and $|f(x) - p(x)| \leq \varepsilon$, we get

$$\left| G_n(f) - G_n(p) \right| \leq \varepsilon(b-a).$$

Combining these we have shown that for any $\varepsilon$, one can always find $n$ such that

$$\left| \int_a^b f(x)dx - G_n(p) \right| \leq 2\varepsilon(b-a).$$

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One of the key ingredients to this proof was that the $w_i$ are all positive. That is not the case for Newton-Cotes methods—for large enough $n$ their quadrature weights can be negative—so no similar result is possible.