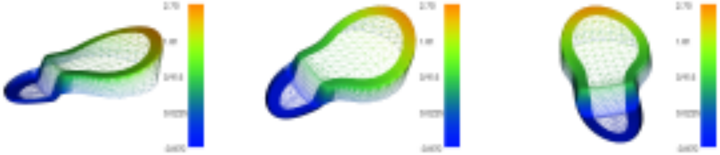
§4 Finite Element Methods

§4.1 Boundary value problems

MA378/531 – Numerical Analysis II ("NA2")

March 2017



(The above images have nothing to do with this lecture. But I like them. The are solutions to a certain PDE computed using a Python-based finite element system called FEniCS.)

In this final section of MA378 we consider numerical schemes for particular ordinary differential equations called

Boundary Value Problems (BVPs).

The main differences between them and the initial value problems (IVPs) of MA385/530 are:

- The equation gives the solution to the differential equation **two** (boundary) points, not just one (initial) point.
- The second derivative of the solution ***always*** appears in the equation. The IVPs we studied usually only had first derivatives.
- We usually think of the independent variable, x, as representing ***space*** rather than time.

For some BVPs it is possible to write down the exact solution (an example is given below). Usually, however, this is not easy, or, indeed, possible. So numerical methods are needed to given *approximate* solutions. Two of the most popular numerical methods are

- **Finite Difference Methods** (FDMs), where one approximates the differential equation, based on Taylor series expansions.
- **Finite Element Methods** (FEMs), where the solution space is approximated, using ideas related to interpolation.

Since **FEMs** are in keeping with the philosophy of this course, we will study them.

More details are in Chapter 14 of the Süli and Mayers.

To formulate a model BVP we first define a differential operator

$$L(u) := -u''(x) + r(x)u(x). \tag{1}$$

Then the general form of a BVP is:

Find a function u, defined on the interval $[a, b]$, such that

$$\begin{aligned} L(u) = f(x) \quad &\text{for} \quad a < x < b, \\ u(a) = 0, u(b) &= 0. \end{aligned} \tag{2}$$

boundary conditions.

We make the assumptions that r and f are continuous and have as many continuous derivatives as we would like. **Furthermore we always have that $r(x) > 0$ for all $x \in [a, b]$.**

Some BVPs are easy to solve by hand. That is, we can write down a solution in terms of elementary functions. That is certainly the case if the functions r and f are constant.

Example

Verify that, for any constants c_0 and c_1,
$u(x) = c_0 e^{2x} + c_1 e^{-2x} + x/4$ is a solution to

$$-u''(x) + 4u(x) = x \quad \text{for} \quad 0 \le x \le 3, \tag{3}$$

(If we have two boundary conditions, we can uniquely determine c_0 and c_1. See Exercise 4.1.1).

Check: $u'(x) = 2c_0 e^{2x} - 2c_1 e^{-2x} + 1/4$
& $u''(x) = 4c_0 e^{2x} + 4c_1 e^{-2x}$

Then $-u'' + 4u = x$, as required.

Unlike the example above, most boundary value problems are difficult or impossible to solve analytically and so, in general, we need approximations. However, it can often be possible to obtain ***qualitative*** information about the solutions.

Lemma (Maximum Principle)

Suppose that L is as defined in (1), with $r(x) > 0$ for all x, and u is a function such that $Lu \geq 0$ on (a, b). If $\boxed{u(a) \geq 0}$, $\boxed{u(b) \geq 0}$. Then $u \geq 0$ for all $x \in [a, b]$.

Proof: Let x^* be such that $u(x^*) = \min_{a \le x \le b} (u(x))$

Suppose that $u(x^*) < 0$. Then $x^* \neq a$ & $x^* \neq b$.

Therefore, u has a local minimum at x^*.

Then $u'(x^*) = 0$, and, moreover, $u''(x^*) \geq 0$.

Thus, $-u''(x^*) \leq 0$ & $r(x^*)u(x^*) < 0$, contradicting $Lu \geq 0$

As we'll now show, this lemma is as useful as it is simple (see also Exercise 4.4). BCs = "Boundary Conditions".

Lemma

There is at most one solution to (2).

Proof. Suppose that u & v both solve (2), ie

$$-u'' + r(x)u(x) = f(x), \quad u(a)=0 \ \& \ u(b)=0.$$

and

$$-v'' + r(x)v(x) = f(x), \quad v(a)=0 \ \& \ v(b)=0.$$

So, if $w = u - v$, then $-w'' + r(x)w = 0$ with $w(a)=0$ & $w(b)=0$. So, from the previous lemma, $w(x) \geq 0$ for all $x \in [a,b]$.

But, similarly, $z = v - u$ solves $-z'' + r(x)z = 0$ + BCs so $z(x) \geq 0$ for all x. But $w = -z$. So $w(x) = 0$ for all $x \in [a,b]$.