§4 Finite Element Methods

§4.2 The variational (weak) formulation

MA378/531 – Numerical Analysis II ("NA2")

March 2017

The key sequence of ideas for FEMs is

(i) First replace the differential equation with an integral equation, using integration by parts to reduce the order of the derivatives. This is called the "*variational*" or "*weak*" form of the equation. (Details are below).

(ii) If we had a candidate for the true solution to the differential equation, we could trial it by substituting it back into the **BVP**.

(iii) But the set of possible solutions is infinitely large, so we can't check them all.

(iv) So we choose a much smaller subset, and look for the solution there. The space we will use is the space of ***piecewise linear splines***, from Section 2.

(v) For every value of the spline that we have to determine, we write down a version of the integral equation that must be satisfied.

(vi) This will give us a linear system of equations to solve.

(vii) A simple but clever idea shows that the approximation we find is the best possible one.

The boundary value problem

$$-u''(x) + r(x)u(x) = f(x) \quad a < x < b,$$

and

$$u(a) = 0, u(b) = 0,$$

equates the terms in u and u'' with a function f that is continuous on (a, b), so we must have that u, u' and u'' are all continuous on the interval (a, b). That is, u belongs to the space of functions $C^2(a, b)$.

So we can state the problem more precisely: *find* $u \in C^2(a, b)$ *such that*

$$-u''(x) + r(x)u(x) = f(x) \qquad \textit{for all } x \textit{ in } (a, b),$$

$$\textit{and } u(a) = u(b) = 0.$$

Where the solution lives

There are (uncountably) many functions in the space $C^2(a,b)$ (we say it is ***infinite dimensional***) – far too many to check one-by-one. So we choose small subset of $C^2(a,b)$ that has only finitely many members. The FEM will let us find the element of that subset that is "closest" to the true solution.

The variational form

Before we give a numerical method for computing an approximate solution to a boundary value problem, we will rewrite it as an integral equation.

Definition

- $C_0^2(a,b)$ to be the space of all functions in $C^2(a,b)$ that are zero at $x=a$ and $x=b$.
- the inner product: $(u,v) := \int_a^b u(x)v(x)dx$; and the induced norm: $\|u\|_2 := (u,u)^{1/2}$,
- and $H_0^1(a,b)$ to be the space of (absolutely) continuous functions on $[a,b]$ such that if $w \in H_0^1(a,b)$ then $w(a)=w(b)=0$ and $\|w'\|_2 < \infty$. (We met similar space before in the section on cubic splines.)

Consider the Boundary Value Problem: *find* $u \in C^2(a,b)$ *such that*

$$\begin{aligned} -u''(x) + r(x)u(x) = f(x) \quad & \textit{on } (a,b), \\ u(a) = u(b) = 0. & \end{aligned} \tag{1}$$

Suppose that we have a solution u to this equation. Then,...

Restrict the set of possible v so that $v(a) = v(b) = 0$ to get another way of stating (1): *Find* u *so that*

$$(u', v') + (ru, v) = (f, v)$$

for all v such that $v(a) = v(b) = 0$.

However, in (1) we required that u be twice differentiable; that seems unnecessary here where we have the much *weaker* requirements that u' exist and be integrable.[1]

[1] Actually, a more correct interpretation of the expression *weak*" formulation is that it does not necessarily involve classical "strong" derivatives, but a more general concept of the weak derivative based on Lebesgue integration. But this is beyond the scope of this course, and so we will ignore this technicality.

Definition (Variational formulation)

The variational/weak formulation of (1) is: *Find* $u \in H_0^1(a,b)$ *such that*

$$A(u,v) = L(v) \quad \textit{for all} \quad v \in H_0^1(a,b). \tag{2}$$

where $\mathcal{A}(\cdot,\cdot)$ is the (symmetric) bilinear functional

$$\mathcal{A}(u,v) := (u',v') + (ru,v).$$

and $L(v)$ is the linear functional

$$L(v) = (f,v).$$

This $\mathcal{A}$ has several important properties. One of these is that, for any function w,

$$A(w, w) \geq 0,$$

and

$$A(w, w) = 0 \Leftrightarrow w = 0.$$

In immediate consequence of this is the following lemma.

Lemma

The variational problem (2) has at most one solution.