§4 Finite Element Methods

§4.3 The FEM

MA378 – Numerical Analysis II (“NA2”)

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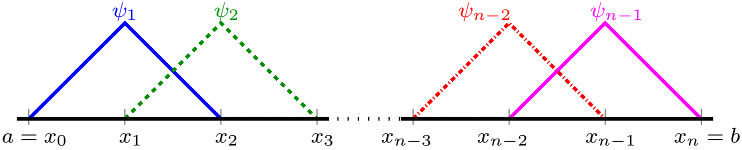


A *space* is a collection of functions. The *dimension* of that space is smallest number of pieces of information that is required to uniquely describe any of its members.

Example

The set of points in $\mathbb{R}^2$. Each member can be written as (x_1, x_2) so we need two pieces of information ("coordinates") to describe the point. So $\mathbb{R}^2$ has dimension 2.

Example

The space of quadratic polynomials (over the reals). Each member can be written as $a + bx + cx^2$. So we need to know a, b and c for each member of the set; the space has dimension 3.

Another way of thinking about this is considering that these space have certain members that can be combined is various ways to give us every other member; they form a *basis* for the space.

Examples of bases for $\mathbb{R}^2$ and $\mathcal{P}^2$:

Example

All the solutions to the differential equation $u''(x) + u(x) = 0$ (note: no boundary conditions) can be written in the form $u(x) = \alpha \sin(x) + \beta \cos(x)$. So $\{\sin(x), \cos(x)\}$ is a basis for this space which is of dimension 2.

All of the spaces mentioned have *finite-dimension*. But consider the space of *all continuous functions on* $[a,b]$. This is an *infinite dimensional space*: we would have to write down an infinite number of values to describe a member uniquely.

Example

The space of functions $C^2(a,b)$ is infinite-dimensional, as is $C_0^2(a,b)$. So too is $H_0^1(a,b)$.

Example

The space of functions that are of the form $g(x) = \gamma(x-a)(x-b)$ for any $\gamma \in \mathbb{R}$ is *finite-dimensional* (because it has dimension 1). But every member of this space also belongs to the infinite dimensional space $H_0^1(a,b)$; it is a *finite-dimensional subspace of* $H_0^1(a,b)$.

Example

To get another **very important** example of a finite dimensional subspace of $H_0^1(a,b)$, first fix a "*mesh*" on $[a,b]$. This is just a set of points $\{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$. Then consider the space of all functions that are *piecewise linear* on this mesh and that vanish at $x = a$ and $x = b$.

This is a a finite-dimensional sub-space of $H_0^1(a,b)$. A reasonable basis for this space would be the hat functions $\{\psi_1, \psi_2, \ldots, \psi_{n-1}\}$ given by

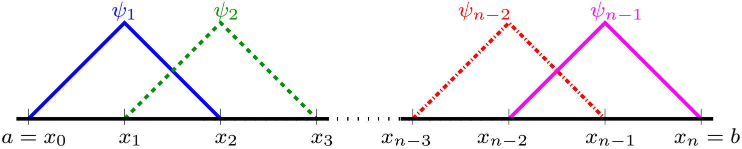
$$\psi_i(x) = \begin{cases} (x - x_{i-1})/h & x_{i-1} \leq x < x_i \\ (x_{i+1} - x)/h & x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

where $h = (b-a)/n$ is the distance between adjacent points.

Then we can write any function u_h as

$$u_h(x) = \lambda_1 \psi_1(x) + \lambda_2 \psi_2(x) + \cdots + \lambda_{n-1} \psi_{n-1}(x).$$

This basis set, shown below, are often called *hat functions* or *Galerkin basis functions*. We met them before in Section 2.1 on piecewise linear interpolation.



Recall that we are trying to solve the problem:

Find $u \in C^2(a,b)$

$$-u''(x) + r(x)u(x) = f(x) \quad \text{on } (a,b), \qquad u(a) = u(b) = 0.$$

We defined $\mathcal{A}(u,v) := (u',v') + (ru,v)$, and wrote the variational form of the DE as:

Find $u \in H_0^1(a,b)$ *such that*

$$\mathcal{A}(u,v) = (f,v) \quad \text{for all} \quad v \in H_0^1(a,b).$$

Imagine we were trying to solve it by taking a function from $H_0^1(a,b)$ and checking if this equation is true for all functions v in $H_0^1(a,b)$. This would take forever because there are an infinite number of candidates.

So instead we restrict our attention to a ***finite-dimensional*** subspace S of $H_0^1(a,b)$. Now we can select a function u_h from S and "put it on trial" by testing it against (all) the functions v_h in S. This leads to the terminology of calling u_h a ***trial*** function and v_h a ***test*** function.

Moreover, it leads to the following method.

Definition (The Finite Element Method)

Let S be the finite dimensional subspace of $H_0^1(a,b)$ made up of the piecewise linear functions on a fixed mesh $a = x_0 < x_1 < \cdots < x_n = b$. Then the *Galerkin Finite Element method* is: *find* $u_h \in S$ *such that*

$$\mathcal{A}(u_h, v_h) = (f, v_h) \qquad \text{for all} \qquad v_h \in S. \tag{1}$$

We now want to look at how to turn Definition 7 into an algorithm (ideally, one that we can implement on a computer).

Let S be the space of piecewise linear functions on the mesh $x_i = a + ih$, where $h = (b-a)/n$. As above, u_h is can be written as

$$u_h(x) = \lambda_1\psi_1(x) + \lambda_2\psi_2(x) + \cdots + \lambda_{n-1}\psi_{n-1}(x).$$

So u_h has $n-1$ unknowns: $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$.

To solve for these, we need $n-1$ equations. To get these, we just choose $n-1$ different (i.e., linearly independent) possible v_h, and substitute into (1).

The most obvious, and (it turns out) sensible, choice for these $n-1$ equations are the $n-1$ hat functions ψ_1, $\psi_2, \ldots, \psi_{n-1}$.

This gives us $n-1$ equations to solve:

$$\mathcal{A}(u_h, \psi_i) = (f, \psi_i) \qquad \text{for } i = 1, \ldots n-1. \tag{2}$$

It is now not too difficult to see that, if we write these equations as a matrix-vector equation, $Ax = F$, then

$$a_{i,j} = \mathcal{A}(\psi_i, \psi_j)$$

Some important observations:

- Any given "hat" function ψ_i is only non-zero on the region $[x_{i-1}, x_{i+1}]$.
- We have to compute

$$a_{ij} = \mathcal{A}(\psi_i, \psi_j) = \int_a^b \psi_i'(x)\psi_j'(x) + r(x)\psi_i(x)\psi_j(x)dx.$$

  But this will be non-zero only if there is overlap between $[x_{i-1}, x_{i+1}]$ and $[x_{j-1}, x_j{+}1]$.
- This gives that $\mathcal{A}(\psi_i, \psi_j) = 0$ if $|i - j| > 1$, and otherwise

$$\mathcal{A}(\psi_i, \psi_j) = \int_{x_{i-1}}^{x_{i+1}} \psi_i'(x)\psi_j'(x) + r(x)\psi_i(x)\psi_j(x)dx.$$

We call such a system ***tridiagonal***. The left-hand side looks like this:

$$\begin{pmatrix} \mathcal{A}(\psi_1,\psi_1) & \mathcal{A}(\psi_2,\psi_1) & 0 & 0 & \cdots & 0 \\ \mathcal{A}(\psi_1,\psi_2) & \mathcal{A}(\psi_2,\psi_2) & \mathcal{A}(\psi_3,\psi_2) & 0 & \cdots & 0 \\ 0 & \mathcal{A}(\psi_3,\psi_2) & \mathcal{A}(\psi_3,\psi_3) & \mathcal{A}(\psi_3,\psi_4) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mathcal{A}(\psi_{n-1},\psi_{n-1} \end{pmatrix}$$

There are two consequences to this:

(i) It takes less effort to set up the linear systems of equations than one might have thought.

(ii) It is relatively easy to solve.

We should also observe the the matrix, A, is **symmetric**.

In general a quadrature rule is used to compute in integrals

$$\int_{x_{i-1}}^{x_{i+1}} r(x)\psi_i(x)\psi_j(x).$$

The Gaussian Quadrature methods are the most popular for this.

Use the FEM on the mesh $\{0,1,2,3\}$ *to find an approximate solution to*

$$-u'' + 3u = x \quad \text{on} \quad (0,3), \quad u(0) = u(3) = 0. \tag{3}$$

Solution: The FEM is: *Find* $u_h(x) \in S$ *such that*

$$\mathcal{A}(u_h, v_h) := (u_h', v_h') + 3(u_h, v_h) = (x, u_h) \qquad \text{for all } v_h(x) \in S.$$

We have that $h = 1$ so let

$$\psi_1(x) = \begin{cases} x & 0 \le x < 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & \text{otherwise,} \end{cases} \qquad \psi_2(x) = \begin{cases} x - 1 & 1 \le x < 2 \\ 3 - x & 2 \le x \le 3 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$u_h(x) = \lambda_1 \psi_1(x) + \lambda_2 \psi_2(x).$$

An example

Our two equations are:

$$(\lambda_1 \psi_1' + \lambda_2 \psi_2', \psi_1') + 3(\lambda_1 \psi_1 + \lambda_2 \psi_2, \psi_1) = (x, \psi_1),$$

$$(\lambda_1 \psi_1' + \lambda_2 \psi_2', \psi_2') + 3(\lambda_1 \psi_1 + \lambda_2 \psi_2, \psi_2) = (x, \psi_2).$$

giving

$$\lambda_1 \left(\int_0^3 \psi_1' \psi_1' dx + 3 \int_0^3 \psi_1 \psi_1 dx \right) + \\ \lambda_2 \left(\int_0^3 \psi_2' \psi_1' dx + 3 \int_0^3 \psi_2 \psi_1 dx \right) = \int_0^3 x \psi_1 dx$$

$$\lambda_1 \left(\int_0^3 \psi_1' \psi_2' dx + 3 \int_0^3 \psi_1 \psi_2 dx \right) + \\ \lambda_2 \left(\int_0^3 \psi_2' \psi_2' dx + 3 \int_0^3 \psi_2 \psi_2 dx \right) = \int_0^3 x \psi_1 dx.$$

We now need to evaluate these integrals. For example, from the 1st equation:

$$\int_0^3 \psi_1'\psi_1' dx = \int_0^1 (1)^2 dx + \int_1^2 (-1)^2 dx = 2,$$

$$\int_0^3 \psi_1\psi_1 dx = \int_0^1 x^2 dx + \int_1^2 (2-x)^2 dx = 2/3,$$

so the coefficient of λ_1 is $2 + 3(2/3) = 4$. Also,

$$\int_0^3 \psi_2'\psi_1' dx = \int_1^2 \psi_2'\psi_1' dx = \int_1^2 (1)(-1) dx = -1,$$

$$\int_0^3 \psi_2\psi_1 dx = \int_1^2 \psi_2\psi_1 dx = \int_1^2 (x-1)(2-x) dx = 1/6.$$

So the coefficient of λ_2 is $-1 + 3(1/6) = -1/2$.

For the right-hand side:

$$\int_0^3 x\psi_1 dx = \int_0^1 (x)(x)dx + \int_1^2 (x)(2-x)dx = 1/3 + 2/3 = 1.$$

Similarly, we can show that

$$\int_0^3 x\psi_2 dx = 2.$$

The final system is

$$4\lambda_1 - \frac{1}{2}\lambda_2 = 1, \qquad -\frac{1}{2}\lambda_1 + 4\lambda_2 = 2.$$

Solving, we get $\lambda_1 = 20/63$ and $\lambda_2 = 34/63$.

The solution is

$$u^h(x) = \frac{20}{63}\psi_1(x) + \frac{34}{63}\psi_2(x).$$