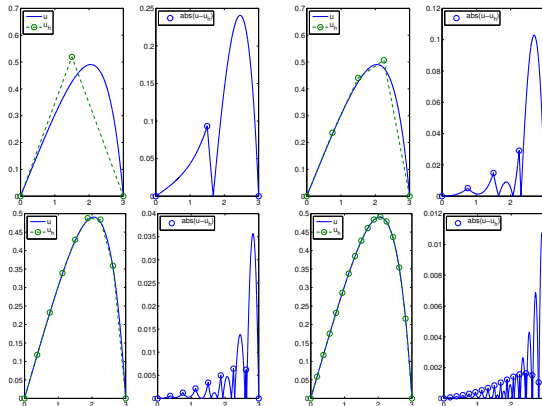


§4 Finite Element Methods

§4.4 Analysis: Cea's Lemma

MA378 – Numerical Analysis II ("NA2")

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Recall that we wrote the differential equation

$$-u''(x) + r(x)u(x) = f(x) \quad \text{on } (a, b), \quad u(a) = u(b) = 0,$$

in a variational form:

Define $A(u, v) := (u', v') + (ru, v)$. Find $u \in H_0^1(a, b)$ such that

$$A(u, v) = (f, v) \quad \text{for all } v \in H_0^1(a, b). \quad (1)$$

We then defined the **FEM**:

Definition (The Finite Element Method)

Let S be a finite dimensional subspace of $H_0^1(a, b)$. The *Galerkin Finite Element method* is: find $u_h \in S$ such that

$$\mathcal{A}(u_h, v_h) = (f, v_h) \quad \text{for all } v_h(x) \in S. \quad (2)$$

We now show that the member of S found by the FEM is the “closest” to the true solution.

Lemma (Cea's Lemma; Thm 14.6 of Süli and Mayers)

Let u be the solution to (1), i.e., the true solution,
and let u_h be the solution to (2), i.e., the FE approximation.

- (i) The difference between the true and approximate solutions is orthogonal to S , i.e.,

$$\mathcal{A}(u - u_h, v_h) = 0 \text{ for all } v_h \in S,$$

and

- (ii) There is no element of S that is closer to u than u_h :

$$\mathcal{A}(u - u_h, u - u_h) = \min_{v_h \in S} \mathcal{A}(u - v_h, u - v_h),$$

First we will prove that

$$\mathcal{A}(u - u_h, v_h) = 0 \text{ for all } v_h \in S,$$

Next we will prove that

$$\mathcal{A}(u - u_h, u - u_h) \leq \mathcal{A}(u - v_h, u - v_h) \text{ for any } v_h \in S.$$

Since $\mathcal{A}(\cdot, \cdot)$ is an inner product (see Definition 3.6.1) it induces a **norm**:

$$|||u||| := \sqrt{\mathcal{A}(u, u)}.$$

So we can write (ii) of Cea's Lemma as

$$|||u - u_h||| \leq |||u - v_h||| \quad \text{for all } v_h \in S.$$

This is as far as we will take the analysis. With a bit more work (and a little Fourier analysis) we could show that

$$\|u - u_h\|_2 \leq Ch^2 \|u''\|_2.$$

That is, the error is proportional to h^2 . We can then further deduce that the method converges:

$$\lim_{h \rightarrow 0} \|u - u_h\|_2 = 0.$$

In place of a rigorous analysis, let us reason as follows. Let l be the piecewise linear interpolant to u as described in Section 2.1. Note that l belongs to S . So, u_h is at least as good an approximation to u as l . That is

$$\|u - u_h\|_2 \leq \|u - l\|_2$$

And Theorem 2.1.3 told us that

$$\|u - l\|_\infty \leq \frac{h^2}{8} \|u''\|_\infty.$$

So, if you believe that

$$\|u - l\|_2 \approx \|u - l\|_\infty,$$

Then it will follow that

$$\|u - u_h\|_2 \lesssim Ch^2,$$

for some constant C . One can also show that

$$|||u - u_h||| \lesssim Ch.$$

Note, however that we have used three different norms here. Therefore much more work would be required to prove a rigorous result. However, we can *demonstrate numerically* that the method converges.

The table opposite shows the maximum error, over all mesh points, in the finite element solution to

$$-u'' + 3u = x \text{ on } (0, 3),$$

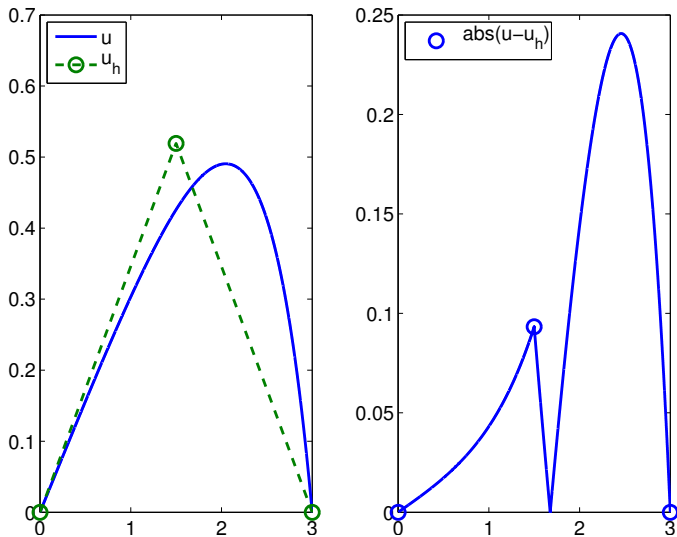
$$u(0) = u(3) = 0.$$

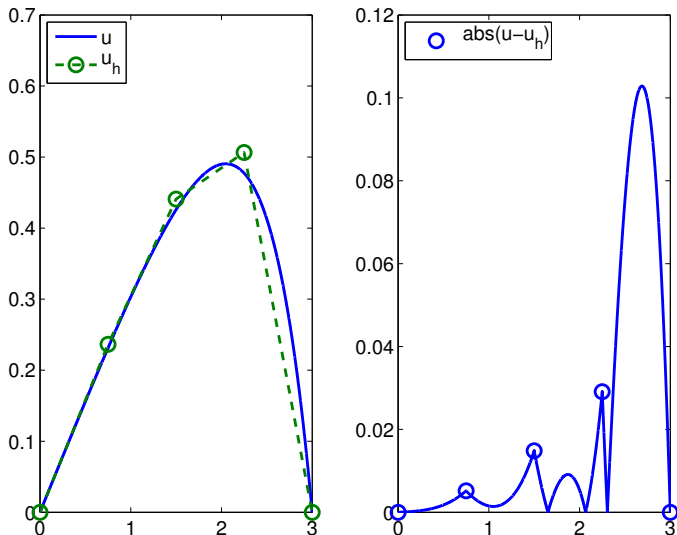
One can see that the error is proportional to n^{-2} (and thus to h^2).

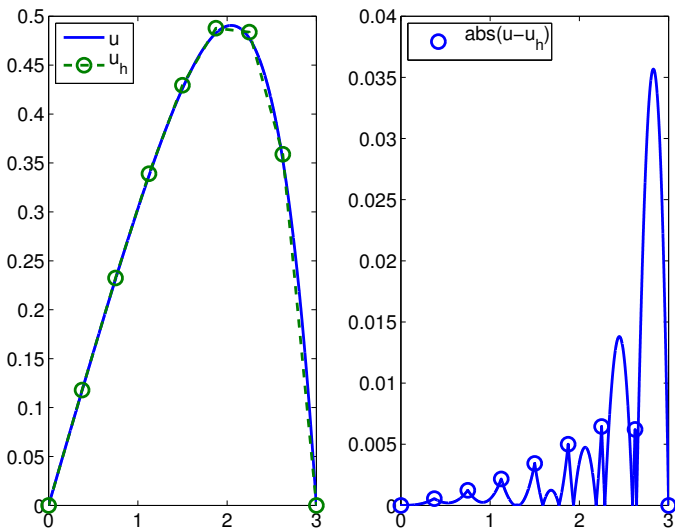
n	$\ u - u_h\ _\infty$
8	6.446e-03
16	1.629e-03
32	4.043e-04
64	1.009e-04
128	2.522e-05
256	6.304e-06
512	1.576e-06
1024	3.940e-07

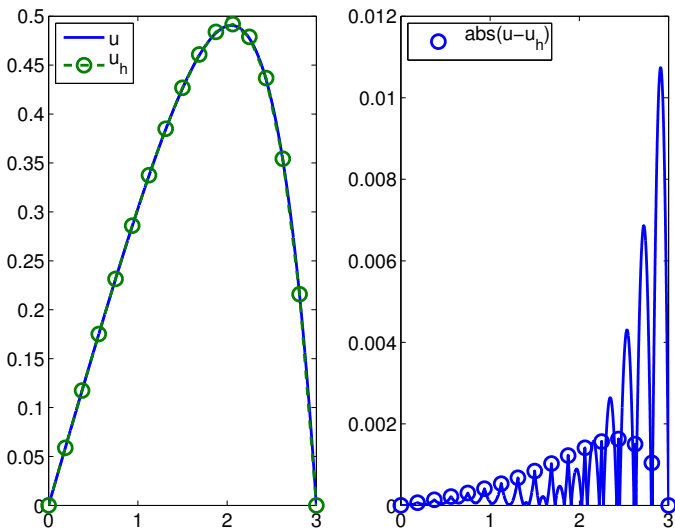
These results were generated by a MATLAB program that can be downloaded from

<http://www.maths.nuigalway.ie/~niall/MA378/fe.m>

Solution and error with $n = 2$ 

Solution and error with $n = 4$ 

Solution and error with $n = 8$ 

Solution and error with $n = 16$ 

There are many aspects of finite element methods that we did not cover, including

1. Finite element methods are ***the most commonly used methods*** for solving problems formulated as differential equations.
2. There are many **other** choices of basis functions given here. One could use cubic splines, or, indeed, higher-order polynomials.
3. When we try to improve the accuracy of the method by reducing h , this is called a h -FEM (and is the most common type).
4. We can also try to improve the accuracy of the method by increasing the order of the polynomials. This is called a p -FEM.
5. The ideas presented here extend to far more general problems. In particular, they work very well for problems in higher dimensions, and on weird-shaped domains.