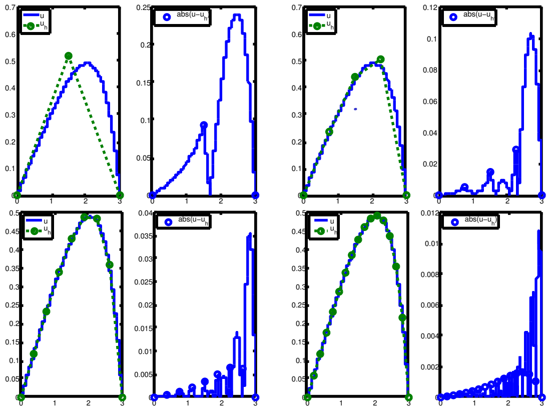
§4 Finite Element Methods

§4.4 Analysis: Cea's Lemma

MA378 – Numerical Analysis II ("NA2")

March 2017



Error analysis Here r & f are given functions,

and $r(x) > 0$

Recall that we wrote the differential equation

$$-u''(x) + r(x)u(x) = f(x) \quad \text{on } (a,b), \qquad u(a) = u(b) = 0,$$

in a variational form:

Define $A(u,v) := (u',v') + (ru,v)$. *Find* $u \in H_0^1(a,b)$ *such that*

$$A(u,v) = (f,v) \quad \text{for all} \quad v \in H_0^1(a,b). \tag{1}$$

We then defined the **FEM**:

Definition (The Finite Element Method)

Let S be a finite dimensional subspace of $H_0^1(a,b)$. The *Galerkin Finite Element method* is: *find* $u_h \in S$ *such that*

$$\mathcal{A}(u_h,v_h) = (f,v_h) \qquad \text{for all} \qquad v_h(x) \in S. \tag{2}$$

We now show that the member of S found by the FEM is the "closest" to the true solution.

Lemma (Cea's Lemma; Thm 14.6 of Süli and Mayers)

Let u be the solution to (1), i.e., the true solution, and let u_h be the solution to (2), i.e, the FE approximation.

(i) *The difference between the true and approximate solutions is orthogonal to S, i.e.,*

$$\mathcal{A}(u - u_h, v_h) = 0 \text{ for all } v_h \in S,$$

and

(ii) *There is no element of S that is closer to u than u_h:*

$$\mathcal{A}(u - u_h, u - u_h) = \min_{v_h \in S} \mathcal{A}(u - v_h, u - v_h),$$

First we will prove that

$$\mathcal{A}(u - u_h, v_h) = 0 \text{ for all } v_h \in S,$$

(This is also called "Galerkin Orthogonality").

Proof We know that

$$A(u, v) = (f, v) \quad \forall v \in H_0^1(a, b).$$

But $S \subseteq H_0^1(a,b)$. So,

$$A(u, v_h) = (f, v_h).$$

$$A(u_h, v_h) = (f, v_h)$$

From the def$^{\underline{n}}$ of FEM for all $v_h \in S$.

Subtracting: $A(u, v_h) - A(u_h, v_h) = 0$

So $A(u - u_h, v_h) = 0$.

Next we will prove that

$$\mathcal{A}(u-u_h, u-u_h) \le \mathcal{A}(u-v_h, u-v_h) \text{ for any } v_h \in S.$$

Proof. For any $v_h \in S$,

$$A(u-v_h, u-v_h) = A((u-u_h)+(u_h-v_h), (u-u_h)+(u_h-v_h))$$

$$= A(u-u_h, u-u_h) + A(u_h-v_h, u-u_h) + A(u-u_h, u_h-v_h) + A(\underbrace{u_h-v_h, u_h-v_h}_{\geq 0})$$

$$\geq A(u-u_h, u-u_h) + 2A(u-u_h, u_h-v_h) \to 0$$

$$= A(u-u_h, u-u_h)$$ Because $u_h \in S$ & $v_h \in S$

So $u_h - v_h \in S$, as (i) applies.

∎

Since $\mathcal{A}(\cdot,\cdot)$ is an inner product (see Definition 3.6.1) it induces a ***norm:***

$$|||u||| := \sqrt{\mathcal{A}(u,u)}.$$

So we can write (ii) of Cea's Lemma as

$$|||u - u_h||| \leq |||u - v_h||| \qquad \text{for all } v_h \in S.$$

This is as far as we will take the analysis. With a bit more work (and a little Fourier analysis) we could show that

$$\|u - u_h\|_2 \leq Ch^2 \|u''\|_2.$$

That is, the error is proportional to h^2. We can then further deduce that the method converges:

$$\lim_{h \to 0} \|u - u_h\|_2 = 0.$$

In place of a rigorous analysis, let us reason as follows. Let l be the piecewise linear interpolant to u as described in Section 2.1. Note that l belongs to S. So, u_h as at least as good an approximation to u as l. That is

$$\|u - u_h\|_2 \leq \|u - l\|_2$$

And Theorem 2.1.3 told us that

$$\|u - l\|_\infty \leq \frac{h^2}{8}\|u''\|_\infty.$$

So, if you believe that

$$\|u - l\|_2 \approx \|u - l\|_\infty,$$

Then it will follow that

$$\|u - u_h\|_2 \lessapprox Ch^2,$$

for some constant C. One can also show that

$$|||u - u_h||| \lessapprox Ch.$$

An example

Note, however that we have used three different norms here. Therefore much more work would be required to prove a rigorous result. However, we can ***demonstrate numerically*** that the method converges.

The table opposite shows the maximum error, over all mesh points, in the finite element solution to

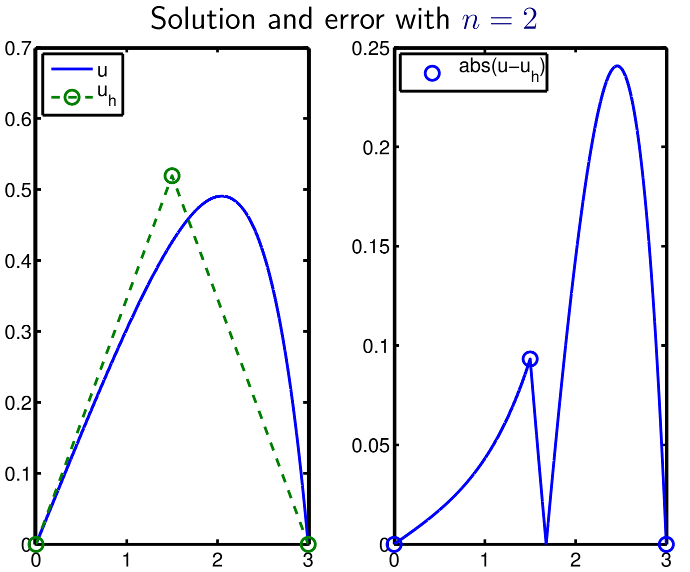
$$-u'' + 3u = x \text{ on } (0,3),$$

$$u(0) = u(3) = 0.$$

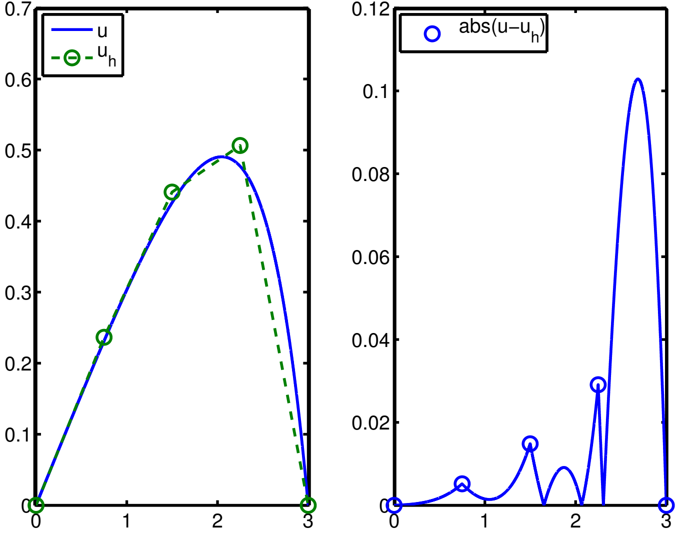
One can see that the error is proportional to n^{-2} (and thus to h^2).

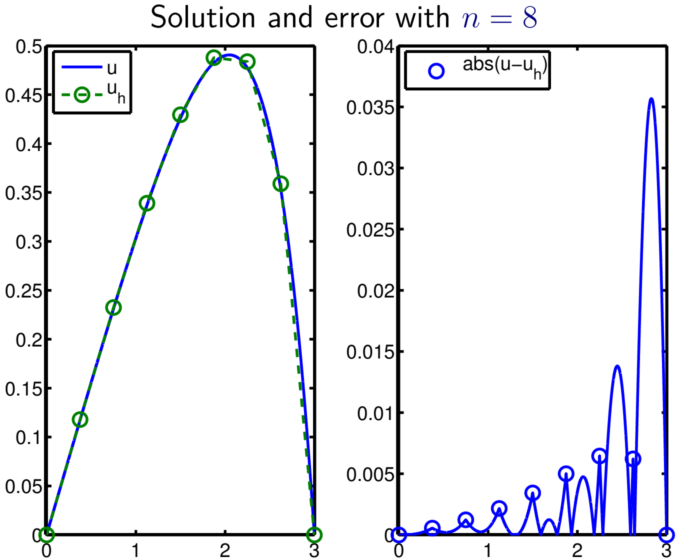
n	$\|u - u_h\|_\infty$
8	6.446e-03
16	1.629e-03
32	4.043e-04
64	1.009e-04
128	2.522e-05
256	6.304e-06
512	1.576e-06
1024	3.940e-07

These results were generated by a MATLAB program that you can download from www.maths.nuigalway.ie/ niall/MA378/fe.m

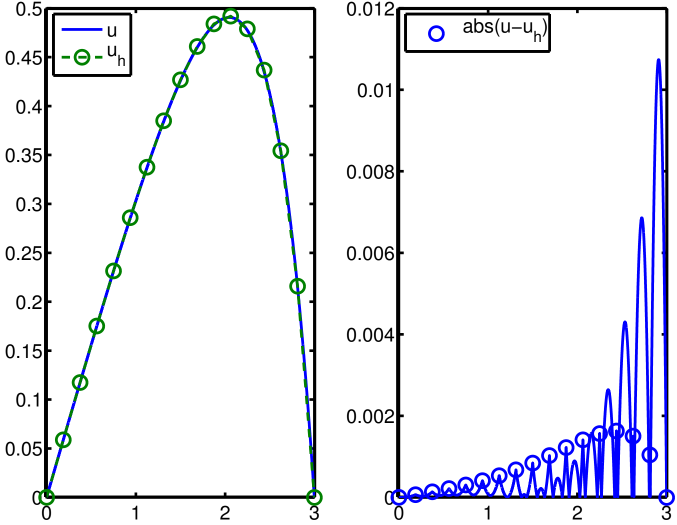

Solution and error with $n = 2$
u
u_h
abs($u-u_h$)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.25
0.2
0.15
0.1
0.05
0
1
2
3

Solution and error with $n = 4$




Solution and error with $n = 8$
u
u_h
abs(u−u_h)
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
1
2
3

Solution and error with $n = 16$



There are many aspects of finite element methods that we did not cover, including

1. Finite element methods are ***the most commonly used methods*** for solving problems formulated as differential equations.
2. There are many **other** choices of basis functions given here. One could used cubic splines, or, indeed, higher-order polynomials.
3. When we try to improve the accuracy of the method by reducing h, this is called a h-FEM (and is the most common type).
4. We can also try to improve the accuracy of the method by increasing the order of the polynomials. This is called a p-FEM.
5. The ideas presented here extend to far more general problems. In particular, they work very well for problems in higher dimensions, and on weird-shaped domains.