Solutions exercises from Assignment 2

3.2 ⋆ *Show that* $\sum_{i=0}^{n} q_i = b - a$.
Solution. Since any Newton Cotes method has precision at least zero, it should be exact for all constant functions. In particular, if $f(x) = 1$, then $Q_n(f) = \int_a^b f(x)dx$. But $Q_n(1) = q_0 + q_1 + \cdots + q_n$. And $\int_a^b 1dx = (b - a)$.

3.8 (iii) ⋆ *Determine the precision of the following schemes for estimating* $\int_0^1 f(x)dx$ *of the following method:*

$$Q(f) = \frac{3}{2}f(\frac{1}{3}) - 2f(\frac{1}{2}) + \frac{3}{2}f(\frac{2}{\mathbf{3}}).$$

Solution: We will determine the precision by seeing which polynomials the method is exact for.

(i) $f(x) \equiv 1$. Then $Q(f) = 3/2 - 2 + 3/2 = 1 = \int_0^1 1dx$.

(ii) $f(x) = x$. Then $Q(f) = (3/2)(1/3) - 2(1/2) + (3/2)(2/3) = 1/2 - 1 + 1 = 1/2 = \int_0^1 xdx$.

(iii) $f(x) = x^2$. Then $Q(f) = (3/2)(1/9) - 2(1/4) + (3/2)(4/9) = 1/6 - 3/6 + 4/6 = 1/3 = \int_0^1 x^2dx$.

(iv) $f(x) = x^3$. Then $Q(f) = (3/2)(1/27) - 2(1/8) + (3/2)(8/27) = 2/(36) - 9/36 + 16/36 = 1/4 = \int_0^1 x^3dx$.

(v) $f(x) = x^4$. Then $Q(f) = (3/2)(1/81) - 2(1/16) + (3/2)(16/81) = 41/216 \neq 1/5 = \int_0^1 x^5dx$.

Therefore, the method has precision $n = 3$.

3.17 ⋆ *Show that it is impossible to choose* $n + 1$ *quadrature points and weights so that the* $n + 1$*-point quadrature rule*

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} w_k f(x_k)$$

has precision $2n + 2$. *Hint: To show the method does not have precision* $2n + 2$, *you just need to give a an example of a single polynomial* p *of degree exactly* $2n + 2$ *for which* $\int_a^b p(x)dx \neq \sum_{k=0}^{n} w_k f(x_k)$.

Solution. Let π_{n+1} be the nodal polynomial associated with the quadrature points, i.e.,

$$\pi_{n+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_n).$$

Let $f(x) = \big(\pi_{n+1}(x)\big)^2$, so f is a polynomial of degree $2n+2$. Also, $f(x) \geqslant 0$ for all x, so $\int_a^b f(x)dx > 0$. However, $f(x_i) = 0$ for all i, so

$$G_n(f) = \sum_{k=0}^{n} w_k f(x_k) = 0.$$

So G_n is *not* exact for this polynomial of degree $2n + 2$. So it does not have precision $2n + 2$.

4.5 ⋆ *Consider the differential equation:*

$$-u''(x) = \exp(x + 1), \text{ on } (0, 2), \quad \text{and} \quad u(0) = u(2) = 0.$$

(i) *State the variational formulation of this differential equation.*
Solution: The variational formulation is: *Find* $u \in H_0^1(0, 2)$ *such that*

$$\int_0^2 u'(x)v'(x)dx = \int_0^2 e^{x+1}v(x)dx \qquad \text{for all } v \in H_0^1(0, 2).$$

(ii) *Show that the solution to the variational problem is unique.*
Solution: Suppose that there are two solutions, u and w. Then, for all $v \in H_0^1(0, 2)$,

$$\int_0^2 u'v'dx = \int_0^2 e^{x+1}v(x)dx \quad \text{and} \quad \int_0^2 w'v'dx = \int_0^2 e^{x+1}v(x)dx$$

Subtracting these two equations, we get that, for all v,

$$\int_0^2 u'v'dx - \int_0^2 w'v'dx = \int_0^2 (u - w)'v'dx = 0.$$

Since this holds for all v, take $v = u - w$, to get that

$$\int_0^2 \Big[\big(u - w\big)'\Big]^2 dx = 0.$$

It follows that $(u-w)'(x) = 0$ for all x. Thus $u(x)-w(x) = C$ for some constants C. But, since $u(0) = w(0) = 0$, we must have $C = 0$. So $u = w$.