MA378 Chapter 1: Polynomial Interpolation

Submit carefully written solutions to Exercises 1.6, 1.8, 1.13, and 1.16

Deadline: 5pm, Friday 10 February.

Your solutions must be clearly written, and neatly presented, and pages should be stapled together. Attach a copy of the Marking Sheet.

Marks will be given for quality and clarity of exposition.

Exercise 1.1. Suppose that $p \in \mathcal{P}_m$ and $q \in \mathcal{P}_n$.

(a) What is the maximum possible degree of $p+q$?

(b) What is the minimum possible degree of $p-q$?

(c) What is the maximum possible degree of pq?

Exercise 1.2. Find out what a *vector space* is. Convince yourself that $\mathcal{P}_n$ is a vector space.

Exercise 1.3. (a) Is it always possible to find a polynomial of degree 1 that interpolates the single point (x_0, y_0)? If so, how many such polynomials are there? Explain your answer.

(b) Is it always possible to find a polynomial of degree 1 that interpolates the two points (x_0, y_0) and (x_1, y_1)? If so, how many such polynomials are there? Explain your answer.

(c) Is it ever possible to find a polynomial of degree 1 that interpolates the three points (x_0, y_0), (x_1, y_1), and (x_2, y_2)? If so, give an example.

Exercise 1.4. Give a proof of Lemma 1.2.2 that is different from the one we did in class.

Exercise 1.5. The general form of the *Vandermonde* Matrix is

$$V_n = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

Its determinant is

$$\det(V_n) = \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i). \qquad (1.0.1)$$

Verify this for the 2×2 and 3×3 cases.

(Note that from formula (1.0.1) we can deduce directly that the PIP has a unique solution *if and only if* the points $x_0, x_1, \ldots, x_n$ are all distinct.)

Aside: Here is a hint for proving that in general

$$\det(V_n) = \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i).$$

First note that $\det(V_n) = \det(V_n^T)$ and now consider the determinant of

$$V_n^T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_0 & x_1 & x_2 & \cdots & x_n \\ x_0^2 & x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_0^n & x_1^n & x_2^n & \cdots & x_n^n \end{pmatrix}.$$

Using elementary row operations (which preserve the determinant), add to each row, the row above it multiplied by $-x_0$, to show that we need to compute the determinant of

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_1 - x_0 & x_2 - x_0 & \cdots & x_n - x_0 \\ 0 & x_1^2 - x_1 x_0 & x_2^2 - x_2 x_0 & \cdots & x_n^2 - x_n x_0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & x_1^n - x_1^{n-1} x_0 & x_2^n - x_2^{n-1} x_0 & \cdots & x_n^n - x_n^{n-1} x_0 \end{pmatrix}.$$

Now try to continue by induction...

Exercise 1.6. $\star$ (MA378 Semester 2 exam, 2015/2016) Let p_3 be the polynomial that interpolates $f(x) = \sin(x/2)$ at the points $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$ and $\mathbf{x_3 = 1}$. Use Cauchy's Theorem to give a bound for the error at $x = 1/2$. (You don't have to give a formula for p_3).[1].

Exercise 1.7. Find the polynomial p_1 that interpolates the function $f(x) = x^3$ at the points $x_0 = 0$ and $x_1 = a$. Find the point $\sigma \in [0, a]$ that maximises $|f(x) - p_1(x)|$, and hence compute $\max_{0 \leqslant x \leqslant a} |f(x) - p_1(x)|$.

Source: Chapter 6 of Süli and Mayers.

Exercise 1.8. $\star$ Show that

$$\sum_{i=0}^{n} L_i(x) = 1 \quad \text{for all } x.$$

[1] An earlier version of this stated, incorrectly, that $x_4 = 1$

Exercise 1.9. Write down the Lagrange Form of p_2, the polynomial of degree 2 that interpolates the points $(0,3)$, $(1,2)$ and $(2,4)$.
Source: Chapter 2 of Stoer and Bulirsch.

Exercise 1.10. Show that all the following represent the same polynomial (usually called the "Chebyshev Polynomial of Degree 3"), $T_3(x) = 4x^3 - 3x$.

(a) Horner form: $\big((4x+0)x-3\big)x+0$.

(b) Lagrange form: $\displaystyle\sum_{k=0}^{3}\Big(\prod_{j=0,j\neq k}^{3}\frac{x-x_j}{x_k-x_j}\Big)(-1)^{k+1}$, where $x_0=-1, x_1=-1/2, x_2=1/2, x_3=1$.

(c) Recurrence relation: $T_0=1$, $T_1=x$, and $T_n = 2xT_{n-1}-T_{n-2}$ for $n=2,3,...$

(d) Trigonometric form: $T_3(x)=\cos\big(3\cos^{-1}(x)\big)$.

Exercise 1.11. Let p_2 be the polynomial of degree 2 that interpolates a function f at the points x_0, x_1 and x_2. If $x_1-x_0=x_2-x_1=h$, show that

$$\max_{x_0\leqslant x\leqslant x_2}|f(x)-p_2(x)| \leqslant \frac{1}{6}\frac{2}{3\sqrt{3}}h^3M_3 = \frac{1}{9\sqrt{3}}h^3M_3.$$

Hint: simplify the calculations by taking $t=x-x_1$, *writing* $(x-x_0)(x-x_1)(x-x_2)$ *in terms of* h *and* t.

Exercise 1.12. Do Exercise 6.3 from Süli and Mayers, *An Introduction to Numerical Analysis*.

Exercise 1.13. ⋆ For *just* the case $n=1$, state and prove an appropriate version of Theorem 1.5.2 (i.e., error in the Hermite interpolant). Use this to find a bound for $\|f-p_3\|_{[x_0,x_1]}$ in terms of f and $h=x_1-x_0$. (Here $\|g\|_{[x_0,x_1]}$ is short-hand for $\max_{x_0\leqslant x\leqslant x_1}|g(x)|$.)

Exercise 1.14. Let $n=2$ and $x_0=-1$, $x_0=1$ and $x_1=1$. Write out the formulae for H_i and K_i for $i=0,1,2$ and give a rough sketch of each of these six functions that shows the value of the function and its derivative at the three interpolation points.

Exercise 1.15. Do Exercise 6.6 from from Süli and Mayers, *An Introduction to Numerical Analysis*.

Exercise 1.16. ⋆ Let L_0, L_1, ..., L_n be the usual Lagrange polynomials for the set of interpolation points $\{x_0,x_1,\dots,x_n\}$. Now define

$$H_i(x)=[L_i(x)]^2(1-2L_i'(x_i)(x-x_i)),$$

and

$$K_i(x)=[L_i(x)]^2(x-x_i).$$

We saw in class that, for $i,k=0,1,\dots n$,

$$H_i(x_k)=\begin{cases}1 & i=k\\ 0 & i\neq k\end{cases} \qquad H_i'(x_k)=0.$$

Show that, for $i,k=0,1,\dots n$,

$$K_i(x_k)=0, \qquad K_i'(x_k)=\begin{cases}1 & i=k\\ 0 & i\neq k\end{cases}$$

Conclude that the solution to the Hermite Polynomial Interpolation Problem is

$$p_{2n+1}(x)=\sum_{i=0}^{n}\big(f(x_i)H_i(x)+f'(x_i)K_i(x)\big).$$

Exercise 1.17. Write down that formula for q_3, the *Hermite* polynomial that interpolates $f(x)=\sin(x/2)$, and its derivative, at the points $x_0=0$ and $x_1=1$. Give an upper bound for $|f(1/2)-q_3(1/2)|$. How does this compare with the bound in Exercise 1.6?

Exercise 1.18. (This exercise is based on Exer 6.5 from Süli and Mayers' *Introduction to Numerical Analysis*). Consider the following problem.

> Take $n+1$ distinct interpolation points $x_0<x_1<\dots<x_n$. Let p_{2n+1} be the polynomial of degree $2n{+}1$ with the property that
>
> $$p_{2n+1}(x_i)=f(x_i),$$
>
> and
>
> $$p''_{2n+1}(x_i)=f''(x_i).$$

In general this problem does *not* have a unique problem.

(i) Explain briefly but carefully why the arguments, based on Rolle's Theorem, used to prove **uniqueness** of solutions to the HPIP, will not work here.

(ii) Show that there is no $p_5(x)$ that solves this problem when

- $x_0=-1$, $x_1=0$, $x_2=1$.
- $f(-1)=1$, $f(0)=0$, $f(1)=1$.
- $f''(-1)=0$, $f''(0)=0$, $f''(1)=0$.