MA378 Exercises from Chapter 3 (Numerical Integration) and Chapter 4 (FEMs for BVPs)

Submit carefully written solutions to Exercises 3.2, 3.8-(iii), 3.17, and 4.5 by **5pm, Monday 27 March, 2017**. *Your solutions must be clearly written, and neatly presented, and pages should be stapled together. Attach a copy of the Marking Sheet.*

Exercise 3.1. Let q_0, q_1, …, q_n be the quadrature weights for the Newton-Cotes rule $Q_n(f)$. Show that $q_i = q_{n-i}$ for $i = 0, \dots n$.

Exercise 3.2. $\star$ Show that $\sum_{i=0}^{n} q_i = b - a$.

Exercise 3.3. Deduce the 4-point Newton-Cotes Rule for estimating the integral $\int_0^1 f(x)dx$:

$$Q_3(f) = q_0 f(x_0) + q_1 f(x_1) + q_2 f(x_2) + q_3 f(x_3).$$

Extend the rule to estimate the integral of functions over $[a, b]$.

Exercise 3.4. Prove the error bound given for the Trapezium rule. That is, show that

$$|\int_a^b f(x)dx - Q_1(f)| := \mathcal{E}_1 \leqslant \frac{(b-a)^3}{12} M_2.$$

Exercise 3.5. Explain clearly, with an example, why in general it is not true that $Q_n(f) \to \displaystyle\int_a^b f(x)dx$ as $n \to \infty$.

Exercise 3.6.

(i) Use Theorem 3.2.2 to deduce an error estimate for the Composite Trapezium Rule (3.7).

(ii) Taking $N = 10$, give an upper bound for the error in the Composite Trapezium Rule when approximating $\int_1^2 \ln(x)dx$.

(iii) What value of n would you have to take to ensure that the error was less that 10^{-5}?

Exercise 3.7.

(i) Deduce the formula for the *composite Simpson's Rule*, and use Theorem 3.3.1 to derive an error estimate.

(ii) What value of N would you have to take to ensure that the error in the estimate of $\int_1^2 \ln(x)dx$ is less that 10^{-6}?

(iii) Denote the (N+1)-point Composite Simpson's Rule by $S_N(f) \approx \displaystyle\int_a^b f(x)dx$. Show that, for sufficiently smooth $f(x)$,

$$\lim_{n\to\infty} S_N(f) = \int_a^b f(x)dx.$$

Exercise 3.8. Determine the precision of the following schemes for estimating $\int_0^1 f(x)dx$.

(i) $Q(f) = f(\frac{1}{2})$.

(ii) $Q(f) = \frac{1}{4}f(0) + \frac{3}{4}f(\frac{2}{3})$.

(iii) $\star$ $Q(f) = \frac{3}{2}f(\frac{1}{3}) - 2f(\frac{1}{2}) + \frac{3}{2}f(\frac{2}{3})$.

Exercise 3.9. Suppose that a quature rule exactly integrates the polynomials 1, x, x^2, …, x^n. Show that it has precision n.

Exercise 3.10. Use a change of variables, as we did with the Trapezium rule, to show that the rule for approximating $\int_0^1 f(x)dx$ is

$$G_1(f) = \frac{1}{2}\Big(f(\frac{1}{2} - \frac{1}{2\sqrt{3}}) + f(\frac{1}{}\frac{1}{+2\sqrt{3}})\Big).$$

More generally, extend the $G_1(f)$ rule in (3.8) to an arbitrary interval $[a, b]$.

Exercise 3.11. Use $G_1(x)$ to estimate $\int_1^2 \ln(x)dx$. How does this compare with the Trapezium and Simpson's Rule?

Exercise 3.12. Derive a 3-point Gaussian Quadrature Rule to estimate $\int_{-1}^1 f(x)dx$. *Hint:* $x_1 = 0$.

Exercise 3.13. Suppose that $(\cdot,\cdot)$ is an inner product. Show that $\|u\| := \sqrt{(u,u)}$ is a norm.

Exercise 3.14. $\mathcal{P}_n$, the space of polynomials of degree (at most) n forms a vector space. Is it true that the space of *monic* polynomials of degree n forms a vector space?

Exercise 3.15. (i) Using the Inner Product

$$(f, g) := \int_{-1}^1 f(x)g(x)dx,$$

find $\widetilde{p}_0(x)$, $\widetilde{p}_1(x)$, $\widetilde{p}_2(x)$ and $\widetilde{p}_3(x)$.

(ii) Find the zeros of $\widetilde{p}_2(x)$ and call them x_0 and x_1. Construct a quadrature rule for $\int_{-1}^1 f(x)dx$ taking these as the quadrature points, and the weights as the integrals to the corresponding Lagrange polynomials. Verify that this is the same rule as given in (3.8).

(iii) Repeat this exercise using the zeros of $\widetilde{p}_3(x)$ as the quadrature points. Verify that the rule you get is the same as Exercise 3.12.

Exercise 3.16. Prove Theorem 3.6.1.

Exercise 3.17. $\star$ Show that it is impossible to choose $n+1$ quadrature points and weights so that the $n+1$-point quadrature rule

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} w_k f(x_k)$$

has precision $2n+2$.

Hint: To show the method does not have precision $2n+2$*, you just need to give a an example of a single polynomial* p *of degree exactly* $2n+2$ *for which* $\int_a^b p(x)dx \neq \sum_{k=0}^{n} w_k f(x_k)$.

Exercise 4.1. Suppose, instead of the differential operator defined in (4.1), we had the more general one:

$$L_q(u) := -u''(x) + q(x)u'(x) + r(x)u(x).$$

Does this L_q also satisfy a maximum principle? If so, provide a proof. If not, give a counter example.

Exercise 4.2. Verify that

$$u(x) = \frac{x}{4} + \frac{3e^6(e^{-2x} - e^{2x})}{4(e^{12} - 1)}$$

is the exact solution to (4.1.1) with the boundary conditions $u(0) = 0$, $u(3) = 0$,

Exercise 4.3. In this section of the course, we'll always assume homogeneous boundary conditions. That is, that $u(x) = 0$ at the boundaries. Suppose the problem we wish to solve is

$$-u''(x) + r(x)u(x) = f(x) \quad u(0) = \alpha, u(1) = \beta.$$

Show how to find a problem which has the same left-hand side as this one, homogeneous boundary conditions, and with a solution that differs from this one only by a known linear function.

Exercise 4.4. Suppose that u solves

$$-u''(x) + r(x)u(x) = f(x) \quad \text{on } (0,1),$$

and $u(0) = u(1) = 0$. Let ρ be such $r(x) \geqslant \rho > 0$, and define

$$C = \max_{0 \leqslant x \leqslant 1} |f(x)|/\rho.$$

Prove that $u(x) \leqslant C$.

Exercise 4.5. $\star$ Consider the differential equation:

$$-u''(x) = \exp(x+1), \text{ on } (0,2), \text{ and } u(0) = u(2) = 0.$$

(i) State the variational formulation of this differential equation.

(ii) Show that the solution to the variational problem is unique.

Exercise 4.6. Show that u_h solves (4.7) *if and only if* it solves (4.6).

Exercise 4.7. Consider the problem:

$$-u''(x) = 9x \qquad u(0) = 0, u(1) = 0.$$

Use the FEM to find an approximate solution on the mesh $\{0, 1/3, 2/3, 1\}$.

Also write down the true solution to this problem.

Exercise 4.8. Suppose we want to use a finite element method to solve

$$-u''(x) + u(x) = 1 \text{ on } (0,1),$$

with $u(0) = u(1) = 0$, using the usual piecewise linear basis functions on on uniform mesh $\{x_0, x_1, \ldots, x_n\}$. Let the resulting linear system is written as the matrix-vector equation $Au_h = F$.

(i) Show that the matrix A is symmetric (i.e. $a_{ij} = a_{ji}$).

(ii) Show that A is tridiagonal (i.e., if $|i-j| > 1$ then $a_{ij} = 0$).

(iii) Derive the formula for the entries of A in terms of h. That is, give an expression for $a_{i,i-1}$, $a_{i,i}$ and $a_{i,i+1}$.

Exercise 4.9. Suppose that we want to solve

$$-u''(x) + u'(x) = 1 \text{ on } (a,b),$$

(a) Write down the system of linear equations that we would have to solve in terms of h.

(b) Does the analysis of Theorem 4.4.1 still hold? That is, can we use a similar argument to show that

$$|||u - u_h||| \leqslant |||u - v_h||| \quad \text{for all } v_h \in S?$$

Exercise 4.10. Show that, for any function $f \in C^2[a,b]$,

$$\|f\|_2 \leqslant \sqrt{b-a}\|f\|_\infty,$$

where

$$\|f\|_2 := \left(\int_a^b (f(x))^2 dx\right)^{1/2} = \sqrt{(f,f)},$$

and

$$\|f\|_\infty := \max_{a \leqslant x \leqslant b} |f(x)|.$$

Exercise 4.10 shows that if we have a bound for $\|f\|_\infty$*, we can get one for* $\|f\|_2$*. However, as the next exercise shows, the converse is not true.*

Exercise 4.11. Show that, given any $\epsilon > 0$, no matter how small, it is possible to construct a function $f \in C^2[a,b]$, for which

$$\|f\|_2 \leqslant \varepsilon$$

but

$$\|f\|_\infty = 1.$$