Solutions to some exercises from Sections 3 and 4, including the homework assignments.

3.1 *Let* q_0, q_1, ..., q_n *be the quadrature weights for the Newton-Cotes rule* $Q_n(f)$. *Show that* $q_i = q_{n-i}$ *for* $i = 0, \ldots n$. *Show that* $q_i = q_{n-i}$ *for* $i = 0, 1, \ldots, n$.

Solution. There are a few possible solutions to this. One is use that $q_i = \int_a^b L_i(x)dx$ where L_i is the usual Lagrange polynomial associated with the point x_i. So we need to show that $\int_a^b L_i(x)dx = \int_a^b L_{n-i}(x)dx$.

Since the quadrature points, x_i are equally spaced with $x_0 = a$ and $x_n = b$, it follows that $x_i = a + ih$ where $h = (b-a)/n$. Similarly, $x_{n-i} = b - ih$. So we see that

$$b + a - x_i = b + a - (a + ih) = b - ih = x_{n-1}.$$

Now we can show that $L_i(x) = L_{n-1}(b + a - x)$ since

- Both $L_i(x)$ and $L_{n-1}(b + a - x)$ are polynomials of degree n;
- $L_i(x_i) = 1$ and $L_{n-1}(b + a - x_i) = L_{n-1}(x_{n-i}) = 1$.
- If $i \neq j$, then $L_i(x_j) = 0$ and $L_{n-1}(b + a - x_j) = L_{n-1}(x_{n-j}) = 0$.

Since there is only one polynomial of degree n with these properties, they must be the same.

Finally, it follows, by employing the change of variables $s = b + a - x$ that

$$q_i = \int_a^b L_i(x)dx = \int_b^a L_{n-i}(b + a - x)dx = -\int_a^b L_{n-i}(s)ds = q_{n-i}.$$

3.2 $\star$ *Show that* $\sum_{i=0}^n q_i = b - a$.

Solution. Since any Newton Cotes method has precision at least zero, it should be exact for all constant functions. In particular, if $f(x) = 1$, then $Q_n(f) = \int_a^b f(x)dx$. But $Q_n(1) = q_0 + q_1 + \cdots + q_n$. And $\int_a^b 1dx = (b - a)$.

3.5 *Explain clearly, with an example, why in general it is not true that* $Q_n(f) \to \displaystyle\int_a^b f(x)dx$ *as* $n \to \infty$.

Solution: Q_n is a Newton-Cotes method: it approximates $\int_a^b f(x)dx$ as $\int_a^b p_n(x)dx$, where p_n is the polynomial of degree n that interpolates f at $n+1$ equally spaced points in $[a, b]$. But suppose that $f(x) = 1/(1 + x^2)$, $a = -5$ and $b = 5$. We know from Section 1, that, as $n- > \infty$, $p_n(x) \to \infty$. Therefore, for this example (Runge's Example), $Q_n(f) \to \infty$.

3.8 (iii) $\star$ *Determine the precision of the following schemes for estimating* $\int_0^1 f(x)dx$ *of the following method:*

$$Q(f) = \frac{3}{2}f(\frac{1}{3}) - 2f(\frac{1}{2}) + \frac{3}{2}f(\frac{2}{\mathbf{3}}).$$

Solution: We will determine the precision by seeing which polynomials the method is exact for.

(i) $f(x) \equiv 1$. Then $Q(f) = 3/2 - 2 + 3/2 = 1 = \int_0^1 1dx$.

(ii) $f(x) = x$. Then $Q(f) = (3/2)(1/3) - 2(1/2) + (3/2)(2/3) = 1/2 - 1 + 1 = 1/2 = \int_0^1 xdx$.

(iii) $f(x) = x^2$. Then $Q(f) = (3/2)(1/9) - 2(1/4) + (3/2)(4/9) = 1/6 - 3/6 + 4/6 = 1/3 = \int_0^1 x^2dx$.

(iv) $f(x) = x^3$. Then $Q(f) = (3/2)(1/27) - 2(1/8) + (3/2)(8/27) = 2/(36) - 9/36 + 16/36 = 1/4 = \int_0^1 x^3dx$.

(v) $f(x) = x^4$. Then $Q(f) = (3/2)(1/81) - 2(1/16) + (3/2)(16/81) = 41/216 \neq 1/5 = \int_0^1 x^5dx$.

Therefore, the method has precision $n = 4$.

3.9 *Suppose that a quadrature rule exactly integrates the polynomials* 1, x, x^2, ..., x^n. *Show that it has precision* n.

Solution: We'll denote the method as $Q(\cdot)$. We are told that $Q(x^k) = \int_a^n x^k$ for $k = 0, 1, \ldots, n$. Let r an an arbitrary polynomial of degree n. It can be written as

$$r(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n,$$

for constants c_0, c_1, ..., c_n. Then

$$\int_a^b r(x)dx = \int_a^b \Big(\sum_{k=0}^n c_kx^k\Big)dx = \sum_{k=0}^n c_k \int_a^b x^kdx = \sum_{k=0}^n c_kQ(x^k) = Q(\sum_{k=0}^n c_kx^k) = Q(r),$$

as required.

3.16 *Prove Theorem 3.6.1.*

Let $x_0, \dots, x_n$ *to be the zeros of* $\widetilde{p}_{n+1}$*, the* $(n+1)$*th polynomial in the sequence of orthogonal monic polynomials* $\{\widetilde{p}_i\}_{i=0}^{\infty}$*. Set*

$$G_n(f) = w_0 f(x_0) + \cdots + w_n f(x_n) \text{ where } w_i = \int_a^b L_i(x)dx. \tag{3.0.1}$$

Then $G_n(f)$ *has precision* $2n+1$.

Solution. We are trying to show that, if f is any polynomial of degree at most $2n+1$, then $G_n(f) = \int_a^b f(x)dx$. Since $\widetilde{p}_{n+1}$ has degree $n+1$ we can write

$$f(x) = \widetilde{p}_{n+1}(x)q(x) + r(x),$$

where q ("quotient") and r ("remainder") are polynomials of degree at most n. Then because $\widetilde{p}_{n+1}$ is orthogonal to *all* polynomials of degree n or less, $\int_a^b \widetilde{p}_{n+1}(x)q(x)dx = 0$. Thus

$$\int_a^b f(x)dx = \int_a^b \widetilde{p}_{n+1}(x)q(x) + r(x)dx = \int_a^b \widetilde{p}_{n+1}(x)q(x)dx + \int_a^b r(x)dx = \int_a^b r(x)dx.$$

Since G_n is an $(n+1)$-point quadrature rule, it is exact for all polynomials of degree n or less. In particular $G_n(r) = \int_a^b r(x)dx$. Also, $\widetilde{p}_{n+1}(x_i) = 0$ for all i. And so,

$$G_n(f) = \sum_{i=0}^{n} w_i f(x_i) = \sum_{i=0}^{n} w_i \big(\widetilde{p}_{n+1}(x_i)q(x_i) + r(x_i)\big) = \sum_{i=0}^{n} w_i r(x_i) = G_n(r) = \int_a^b r(x)dx = \int_a^b f(x)dx.$$

3.17 $\star$ *Show that it is impossible to choose* $n+1$ *quadrature points and weights so that the* $n+1$*-point quadrature rule*

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} w_k f(x_k)$$

has precision $2n+2$*. Hint: To show the method does not have precision* $2n+2$*, you just need to give a an example of a single polynomial* p *of degree exactly* $2n+2$ *for which* $\int_a^b p(x)dx \neq \sum_{k=0}^{n} w_k f(x_k)$.

Solution. Let π_{n+1} be the nodal polynomial associated with the quadrature points, i.e.,

$$\pi_{n+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_n).$$

Let $f(x) = \big(\pi_{n+1}(x)\big)^2$, so f is a polynomial of degree $2n+2$. Also, $f(x) \geqslant 0$ for all x, so $\int_a^b f(x)dx > 0$. However, $f(x_i) = 0$ for all i, so

$$G_n(f) = \sum_{k=0}^{n} w_k f(x_k) = 0.$$

So G_n is *not* exact for this polynomial of degree $2n+2$. So it does not have precision $2n+2$.

4.1 *Does the differential operator*

$$L_q(u) := -u''(x) + q(x)u'(x) + r(x)u(x).$$

satisfy a maximum principle? This is, is it true that, if $L_q u(x) \geqslant 0$ *for all* $x \in (a,b)$*, and* $u(a) \geqslant 0$*,* $u(b) \geqslant 0$*, then then* $u \geqslant 0$ *for all* $x \in [a,b]$*? If so, provide a proof. If not, give a counter example.* Solution: **Yes**, L_q *does* satisfy a maximum principle.

Proof: let $x^\star \in [a,b]$ be a point where u takes its minimum value. That is

$$u(x^\star) = \min_{a \leqslant x \leqslant b} u(x).$$

We are trying to show that $u(x^\star) \geqslant 0$.

Suppose that $L_q u(x) \geqslant 0$ for all $x \in (a,b)$, and $u(a) \geqslant 0$, $u(b) \geqslant 0$, but that $u(x^\star) < 0$. In that case, $x^\star \neq a$ and $x^\star \neq b$, so u as a *local minimum* at $x^\star$. Then, from elementary calculus, $u'(x^\star) = 0$ and $u''(x^\star) \geqslant 0$. It follows that

$$\underbrace{-u''(x)}_{\leqslant 0} + \underbrace{q(x)u'(x)}_{=0} + \underbrace{r(x)}_{>0}\,\underbrace{u(x)}_{<0} < 0,$$

which contradicts the assumption that $L_q u(x) \geqslant 0$ for all $x \in (a,b)$. So

$$\min_{a \leqslant x \leqslant b} u(x) \geqslant 0.$$

That is, $u(x) \geqslant 0$ for all $x \in [a,b]$, as required.

4.4 *Suppose that* u *solves*

$$-u''(x) + r(x)u(x) = f(x) \quad \text{on } (0,1),$$

and $u(0) = u(1) = 0$*. Let* ρ *be such* $r(x) \geqslant \rho > 0$*, and define*

$$C = \max_{0 \leqslant x \leqslant 1} |f(x)|/\rho.$$

Prove that $u(x) \leqslant C$.

Proof: Let v be the function $v(x) = C - u(x)$. Note that $C > 0$, so $v(0) = v(1) = C > 0$. Also $-v''(x) + r(x)v(x) > 0$, because

$$\begin{aligned} -v''(x) + r(x)v(x) &= -(C-u)''(x) + r(x)(C - u(x)) = \underbrace{u''(x) - r(x)u(x)}_{=-f(x)} + r(x)C \\ &= -f(x) + \frac{r(x)}{\rho} \max_{0 \leqslant x \leqslant 1} |f(x)| \geqslant -f(x) + \max_{0 \leqslant x \leqslant 1} |f(x)| > 0, \end{aligned}$$

where here we have used that $C''(x) \equiv 0$, and $r(x)/\rho \geqslant 1$. So, applying our maximum principle, we get that $v \geqslant 0$. So $u \leqslant C$, as required.

4.5 $\star$ *Consider the differential equation:*

$$-u''(x) = \exp(x+1), \text{ on } (0,2), \quad \text{and} \quad u(0) = u(2) = 0.$$

(i) *State the variational formulation of this differential equation.*

Solution: The variational formulation is: *Find* $u \in H_0^1(0,2)$ *such that*

$$\int_0^2 u'(x)v'(x)dx = \int_0^2 e^{x+1} v(x)dx \qquad \text{for all } v \in H_0^1(0,2).$$

(ii) *Show that the solution to the variational problem is unique.*

Solution: Suppose that there are two solutions, u and w. Then, for all $v \in H_0^1(0,2)$,

$$\int_0^2 u'v'dx = \int_0^2 e^{x+1}v(x)dx \quad \text{and} \quad \int_0^2 w'v'dx = \int_0^2 e^{x+1}v(x)dx$$

Subtracting these two equations, we get that, for all v,

$$\int_0^2 u'v'dx - \int_0^2 w'v'dx = \int_0^2 (u-w)'v'dx = 0.$$

Since this holds for all v, take $v = u - w$, to get that

$$\int_0^2 \Big[(u-w)'\Big]^2 dx = 0.$$

It follows that $(u-w)'(x) = 0$ for all x. Thus $u(x) - w(x) = C$ for some constants C. But, since $u(0) = w(0) = 0$, we must have $C = 0$. So $u = w$.

4.8 *Suppose we want to use a finite element method to solve*

$$-u''(x) + u(x) = 1 \text{ on } (0,1),$$

with $u(0) = u(1) = 0$*, using the usual piecewise linear basis functions on on uniform mesh* $\{x_0, x_1, \dots, x_n\}$*. Let the resulting linear system is written as the matrix-vector equation* $Au_h = F$.

(i) *Show that the matrix* A *is symmetric (i.e.* $a_{ij} = a_{ji}$*).*

Solution: the entries in A are $a_{ij} = \mathcal{A}(\psi_j, \psi_i)$, where $\mathcal{A}$ is the bilinear form:

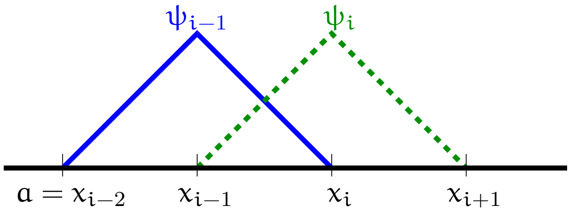
$$\mathcal{A}(u,v) = \int_0^1 u'(x)v'(x) + u(x)v(x)dx.$$

But $\mathcal{A}(v,u) = \int_0^1 v'(x)u'(x) + v(x)u(x)dx = \mathcal{A}(u,v)$. So $a_{ij} = \mathcal{A}(\psi_j, \psi_i) = \mathcal{A}(\psi_i, \psi_j) = a_{ji}$.

(ii) *Show that A is tridiagonal (i.e., if $|i-j|>1$ then $a_{ij}=0$).*
Solution: The basis functions for the method, $\{\psi_1, \psi_2, \dots, \psi_{n-1}\}$, have the formulae

$$\psi_i(x) = \begin{cases} (x - x_{i-1})/h & x_{i-1} \leqslant x < x_i \\ (x_{i+1} - x)/h & x_i \leqslant x \leqslant x_{i+1} \\ 0 & \text{otherwise,} \end{cases} \tag{3.0.2}$$

where $h = x_i - x_{i-1}$, pictured below.



Note that $\psi_i(x)$ is only non-zero on $[x_{i-1}, x_{i+1}$. Therefore, if $i > j+1$ or $j > i+1$, then $\psi_i(x)\psi_j(x) = 0$ for all x. As mentioned above,

$$a_{i,j} = \mathcal{A}(\psi_i, \psi_j) = \int_0^1 \psi_i'(x)\psi_j'(x) + \psi_i(x)\psi)j(x)dx$$

So, if $|i-j|>1$, then $a_{ij} = 0$.

(iii) *Derive the formula for the entries of A in terms of h. That is, give an expression for $a_{i,i-1}$, $a_{i,i}$ and $a_{i,i+1}$.*

Solution: First we'll compute $a_{i,i-1} = \mathcal{A}(\psi_i, \psi_{i-1})$.

$$\mathcal{A}(\psi_i, \psi_{i-1}) = \int_0^1 \psi_i'(x)\psi_{i-1}'(x) + \psi_i(x)\psi_{i-1}(x)dx = \int_{x_{i-1}}^{x_i} \psi_i'(x)\psi_{i-1}'(x)dx + \int_{x_{i-1}}^{x_i} \psi_i(x)\psi_{i-1}(x)dx,$$

because, as illustrated above, the only interval where ψ_{i-1} and ψ_i are both non-zero is $[x_{i-1}, x_i]$.
From (3.0.2),

$$\psi_i'(x) = \begin{cases} 1/h & x_{i-1} \leqslant x < x_i \\ -1/h & x_i \leqslant x \leqslant x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_{x_{i-1}}^{x_i} \psi_i'(x)\psi_{i-1}'(x)dx = \int_{x_{i-1}}^{x_i} \frac{1}{h}\frac{-1}{h}dx = -\frac{x}{h^2}\Big|_{x_{i-1}}^{x_i} = -\frac{x_i - x_{i-1}}{h^2} = -\frac{1}{h}.$$

For $\int_{x_{i-1}}^{x_i} \psi_i'(x)\psi_{i-1}'(x)dx$, we can simplify a little by setting $s = x - x_{i-1}$. Then $x_i - x = x_{i-1} + h - x = h - s$.

$$\int_{x_{i-1}}^{x_i} \psi_i(x)\psi_{i-1}(x)dx = \int_{x_{i-1}}^{x_i} \frac{x - x_i}{h}\frac{x_{i-1} - x}{h}dx = \int_0^h \frac{s}{h}\frac{h-s}{h}ds = \frac{1}{h^2}\big(-\frac{1}{3}s^3 + \frac{1}{2}hs^2\big)\big|_0^h = \frac{h}{6}.$$

So $\mathbf{a_{i,i-1} = -1/h + h/6 = a_{i,i+1}}$.
Next we'll calculate $a_{i,i}$:

$$a_{ii} = \mathcal{A}(\psi_i, \psi_{i-1}) = \int_0^1 \psi_i'(x)\psi_i'(x) + \psi_i(x)\psi_i(x)dx = \int_{x_{i-1}}^{x_{i+1}} \big(\psi_i'(x)\big)^2 + \big(\psi_i(x)\big)^2 dx$$

First,

$$\int_{x_{i-1}}^{x_{i+1}} \big(\psi_i'(x)\big)^2 dx = \int_{x_{i-1}}^{x_i} \big(\frac{1}{h}\big)^2 dx + \int_{x_i}^{x_{i+1}} \big(-\frac{1}{h}\big)^2 dx = \frac{2}{h}.$$

For the $\int_{x_{i-1}}^{x_{i+1}} \big(\psi_i'(x)\big)^2 dx$ term, we'll again simplify by setting $s = x - x_{i-1}$. This gives

$$\int_{x_{i-1}}^{x_{i+1}} \big(\psi_i(x)\big)^2 dx = \int_0^h \big(\frac{s}{h}\big)^2 ds + \int_h^{2h} \big(\frac{h-s}{h}\big)^2 ds = \frac{s^3}{3h^2}\Big|_0^h + \frac{-(h-s)^3}{3h^2}\Big|_h^{2h} = \frac{h}{3} + \frac{h}{3}$$

So $\mathbf{a_{i,i} = 2/h + 4h/6}$.