# Answers to some questions from **Chapter 1**, including homework exercises.

**[Exer 1.6]** $\star$ *Let $p_3$ be the polynomial that interpolates $f(x) = \sin(x/2)$ at the points $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$ and $x_3 = 1$. Use Cauchy's Theorem to give a bound for the error at $x = 1/2$. (You don't have to give a formula for $p_3$).*
**Solution:** From Cauchy's theorem, we know that

$$|f(\frac{1}{2}) - p_4(\frac{1}{2})| \le \max_{0\le\tau\le1} |f^{(4)}(\tau)|\frac{1}{4!}|\pi_4(\frac{1}{2})|,$$

where $\pi_4(x) = x(x-1/3)(x-2/3)(x-1)$. Since $f^{(4)}(\tau) = \sin(\tau/2)/16$, and $|\sin(x)| \le 1$ for all $x$, we can say that $\max_{0\le x\le1} |f^{(4)}(\tau)| \le 1/16$. Also, $\pi_4(1/2) = 1/144$. So this gives that

$$|f(\frac{1}{2}) - p_4(\frac{1}{2})| \le \frac{1}{24\times16\times144} = \frac{1}{55296} = \mathbf{1.808\times10^{-5}}.$$

You can get a slightly sharper bound by using that $\max_{0\le x\le1} |\sin(x/2)| = \sin(1/2) \approx 0.4794$. With this we can get $|f(\frac{1}{2}) - p_4(\frac{1}{2})| \le \mathbf{8.6702\times10^{-6}}$.
(The actually error is $4.46386\times10^{-6}$).

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*Common error:* Several people took Cauchy's theorem as saying $|f(\frac{1}{2}) - p_4(\frac{1}{2})| \le |f^{(4)}(\frac{1}{2})|\frac{1}{4!}|\pi_4(\frac{1}{2})|$. In fact, it is

$$|f(\frac{1}{2}) - p_4(\frac{1}{2})| \le |f^{(4)}(\tau)|\frac{1}{4!}|\pi_4(\frac{1}{2})|, \text{ for some } \tau \in (x_0, x_3).$$

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*Extra:* You were *not* asked to give an error bound for *all* $x$. But if you were, the tricky bit is finding $\max_{0\le x\le1} |x(x-1/3)(x-2/3)(x-1)| = |x^4 - 2x^3 + (11/9)x^2 - (2/9)x|$, which is shown in Figure 1.

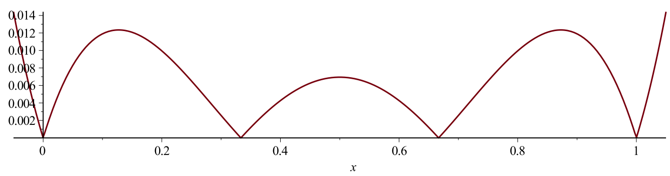


Figure 1: $|\pi_4(x)| = |x(x-1/3)(x-2/3)(x-1)|$

One way of doing that is differentiating $\pi_4$, and finding its zeros. That is, solve

$$4x^3 - 6x^2 + (22/9)x - 2/9 = 0.$$

You can make that a little simpler by using that you already know one of them is at $x = 1/2$. Factoring that out gives you a quadratic: $4x^2 - 4x + 4/9$. Its roots are at $1/2 \pm \sqrt{5}/6$. We can then find that the maximum value of $\pi_4$ on $[0, 1]$ is $1/81$.

**[Exer 1.8]** $\star$ Show that $\sum_{i=0}^{n} L_i(x) = 1$ for all $x$.

**Solution (1):** Define the function $p$ as $p(x) = 1 - \big(L_0(x) + L_1(x) + \cdots + L_n(x)\big)$. Since the $L_i$ are all polynomials of degree $n$, it follows that $p$ is a polynomial, and has degree (at most) $n$. However, for each $k = 0, 1, \ldots, n$,

$$\sum_{i=0}^{n} L_i(x_k) = L_k(x_k) = 1 \qquad \text{since} \qquad L_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \ne k. \end{cases}$$

It follows that $p(x_k) = 0$ for $k = 0, 1, \ldots, n$. That is, $p$ is a polynomial of degree $n$ with $n+1$ zeros. By Lemma 1.2.2, $p(x) = 0$ for all $x$. Thus $\sum_{i=0}^{n} L_i(x) = 1$ for all $x$.

**Solution (2):** Let $f$ be the constant function with $f(x) = 1$ for all $x$. Let $p$ be the polynomial that interpolates $f$ at $n+1$ distinct points, $x_0$, $x_1$, $\ldots$, $x_n$. Then

$$p(x) = \sum_{i=0}^{n} f(x_i)L_i(x) = \sum_{i=0}^{n} L_i(x).$$

Since $f$ is a polynomial of degree (at most) $n$, it is its own, unique, polynomial interpolant. So $p(x) = f(x) = 1$ for all $x$.

**[Exer 1.13]** $\star$ For *just* the case $n = 1$, state and prove an appropriate version of Theorem 1.5.2 (i.e., error in the Hermite interpolant). Use this to find a bound for $\|f - p_3\|_{[x_0,x_1]}$ in terms of $f$ and $h = x_1 - x_0$. (Here $\|g\|_{[x_0,x_1]}$ is short-hand for $\max_{x_0 \le x \le x_1} |g(x)|$.)

**Solution:** For the first part, taking $n = 1$ in Theorem 1.5.2 gives

$$f(x) - q_3(x) = \frac{f^{(4)}(\tau)}{4!}[\pi_2(x)]^2. \tag{1}$$

We prove this as follows. First note that if $x = x_0$ or $x = x_1$ then (1) must be true since both sides of it are zero. For *any* other $x$ in $[x_0, x_1]$, define the auxiliary function

$$g(t) = f(t) - q_3(t) - \frac{f(x) - q_3(x)}{\big(\pi_2(x)\big)^2}\big(\pi_2(t)\big)^2.$$

Then

$$g(x_0) = f(x_0) - q_3(x_0) - \frac{f(x) - q_3(x)}{\big(\pi_2(x)\big)^2}\big(\pi_2(x_0)\big)^2 = 0,$$

since $f(x_0) = q_3(x_0)$ and $\pi_2(x_0) = 0$. Similarly $g(x_1) = 0$. Also,

$$g(x) = f(x) - q_3(x) - \frac{f(x) - q_3(x)}{\big(\pi_2(x)\big)^2}\big(\pi_2(x)\big)^2 = 0.$$

So $g(x)$ has three distinct zeros: at $x_0$, $x$ and $x_1$. By Rolle's theorem, $g'(x)$ has two zeros; we'll call them $\tau_1$ and $\tau_2$ and note that $x_0 < \tau_1 < x$, and $x < \tau < x_1$. Also,

$$g'(t) = f'(t) - q_3'(t) - \frac{f(x) - q_3(x)}{\big(\pi_2(x)\big)^2}2\pi_2(t)\pi_2'(t).$$

So $g'(x_0) = 0$ since $f'(x_0) - q_3'(x_0)$ and $\pi_2(x_0) = 0$. Similarly $g'(x_1) = 0$.
We now know that $g'$ has four distinct zeros in $[x_0, x_1]$; these are $x_0$, $\tau_1$, $x$, $\tau_2$ and $x_1$. By repeated use of Rolle's theorem, there is a point $\tau \in [x_0, x_1]$ such that $g^{(4)}(\tau) = 0$. Using that

$$\frac{d^4}{dx^4}q_3(x) \equiv 0 \quad \text{and} \quad \frac{d^4}{dx^4}\big(\pi_2(x)\big)^2 = 24,$$

a little rearrangement of $g^{(4)}(\tau) = 0$ completes the proof.
For the second part, from (1),

$$\max_{x_0 \le x \le x_1} |f(x) - q_3(x)| \le \frac{1}{24} \max_{x_0 < x < x_1} |f^{(4)}(\tau)| \max_{x_0 < x < x_1} \big((x - x_0)(x - x_1)\big)^2.$$

To find the value of $\max_{x_0 < x < x_1} \big((x - x_0)(x - x_1)\big)^2$, note that the maximum of $|(x - x_0)(x - x_1)|$ occurs at $x = (x_0 + x_1)/2$. So

$$\begin{aligned}\max_{x_0 < x < x_1} |\big((x - x_0)(x - x_1)\big)^2| &= \Big(\big((x_0 + x_1)/2 - x_0\big)\big((x_0 + x_1)/2 - x_1\big)\Big)^2 \\ &= \Big(\big((-x_0 + x_1)/2\big)\big((x_0 - x_1)/2\big)\Big)^2 = \Big(\big(-h/2\big)\big(h/2\big)\Big)^2 = h^4/16\end{aligned}$$

To conclude, we have shown that

$$\max_{x_0 \le x \le x_1} |f(x) - q_3(x)| \le \frac{h^4}{384} \max_{x_0 < x < x_1} |f^{(4)}(\tau)|$$

**[Exer 1.16]** $\star$ *Let $L_0$, $L_1$, …, $L_n$ be the usual Lagrange polynomials for the a set of interpolation points $\{x_0, x_1, \ldots, x_n\}$. Now define*

$$H_i(x) = [L_i(x)]^2(1 - 2L_i'(x_i)(x - x_i)), \qquad \textit{and} \qquad K_i(x) = [L_i(x)]^2(x - x_i). \tag{2}$$

*We saw in class that, for $i, k = 0, 1, \ldots n$,*

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \ne k \end{cases} \qquad\qquad H_i'(x_k) = 0. \tag{3a}$$

*Show that, for $i, k = 0, 1, \ldots n$,*

$$K_i(x_k) = 0, \qquad K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \tag{3b}$$

*Conclude that the solution to the Hermite Polynomial Interpolation Problem is*

$$p_{2n+1}(x) = \sum_{i=0}^{n} \big( f(x_i) H_i(x) + f'(x_i) K_i(x) \big). \tag{4}$$

**Solution:** We know that

$$L_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}. \tag{5}$$

Since $K_i(x) = [L_i(x)]^2 (x - x_i)$, we can see that $K_i(x_k) = 0$ for all $k$ since,

- if $i = k$, then $x_k - x_i = 0$.
- if $i \neq k$, then $L_i(x_k) = 0$.

Next, differentiate $K_i$ to get $K_i'(x) = 2L_i(x)L_i'(x)(x - x_i) + L_i(x)^2$. But using the same reasoning as above, $L_i(x_k)(x_k - x_i) = 0$ for all $k$. So $K_i'(x_k) = L_i(x_k)^2$. From (5), we get the desired result.
To conclude that the formula in (4) solves the HPIP, we must show that

(i) $p_{2n+1}$ is indeed a polynomial of degree $n$,

(ii) $p_{2n+1}(x_k) = f(x_k)$ for $k = 0, 1, \ldots, n$,

(iii) $p_{2n+1}'(x_k) = f'(x_k)$ for $k = 0, 1, \ldots, n$.

For (i), note that $L_i(x)$ is a polynomial of degree $n$. So $(L_i(x))^2$ is a polynomial of degree $2n$. It follows from (2) that $H_i$ and $K_i$ are polynomials of degree $2n + 1$. (To see this, you might have to note that $L_i'(x_i)$ is a constant). So, because $p_{2n+1}$ is the sum of the $H_i$ and $K_i$ terms, it too is a polynomial of degree $2n + 1$.
For (ii) we use that

$$\begin{aligned} p_{2n+1}(x_k) &= \sum_{i=0}^{n} \big( f(x_i) H_i(x_k) + f'(x_i) K_i(x_k) \big) \\ &= \sum_{i=0}^{n} f(x_i) H_i(x_k) && \text{since } K_i(x_k) = 0 \\ &= f(x_k) && \text{since } H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k. \end{cases} \end{aligned}$$

For (iii) we use that

$$\begin{aligned} p_{2n+1}'(x_k) &= \sum_{i=0}^{n} \big( f(x_i) H_i'(x_k) + f'(x_i) K_i'(x_k) \big) \\ &= \sum_{i=0}^{n} f'(x_i) K_i(x_k) && \text{since } H_i'(x_k) = 0 \\ &= f'(x_k) && \text{since } K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k. \end{cases} \end{aligned}$$

**[Exer 1.17]** *Write down that formula for $q_3$, the* Hermite *polynomial that interpolates $f(x) = \sin(x/2)$, and its derivative, at the points $x_0 = 0$ and $x_1 = 1$.*
**Solution:** For this problem,

$$L_0(x) = 1 - x \quad \text{and} \quad L_1(x) = x.$$

So

$$H_0(x) = (1 - x)^2 (1 + 2x) \quad \text{and} \quad H_1(x) = x^2 (3 - 2x),$$

and

$$K_0(x) = (1 - x)^2 x \quad \text{and} \quad K_1(x) = x^2 (x - 1).$$

Recalling (4),

$$p_3(x) = \sin(1/2)H_1(x) + (1/2)K_0(x) + (1/2)\cos(1/2)K_1(x).$$

*Given an upper bound for $|f(1/2) - q_3(1/2)|$. How does this compare with the bound in Q1?*
**Solution:** We know from (1) that

$$|f(1/2) - q_3(1/2)| = \frac{1}{4!} \max_{x_0 \le x \le x_0} |f^{(4)}(x)| \max_{x_0 \le x \le x_1} [\pi_2(x)]^2.$$

As seen in Q1, $|f^{(4)}(x)| \le 1/16$ for any $x$. We know that the maximum of $|\pi_2(x)|$ is at $x = 1/2$. So $\max_{x_0 \le x \le x_1} [\pi_2(x)]^2 = 1/16$. From this we can get that $|f(1/2) - q_3(1/2)| \le 1/6144 \approx 1.628 \times 10^{-4}$. So this is *less* accurate than polynomial interpolation on 4 points, by (roughly) a factor of 10.
**[Exer 1.18]** *(This exercise is based on Exer 6.5 from Süli and Mayers'* Introduction to Numerical Analysis*). Consider the following problem.*

> *Take $n+1$ distinct interpolation points $x_0 < x_1 < \cdots < x_n$. Let $p_{2n+1}$ be the polynomial of degree $2n+1$ with the property that*
>
> $$p_{2n+1}(x_i) = f(x_i), \quad \text{and} \quad p''_{2n+1}(x_i) = f''(x_i).$$

*In general this problem does* not *have a unique problem.*

(i) *Explain briefly but carefully why the arguments, based on Rolle's Theorem, used to prove* ***uniqueness*** *of solutions to the HPIP, will not work here.*
We start as usual, and assume there are two solutions, $p$ and $q$, so that

$$p(x_i) = q(x_i) = f(x_i) \qquad p''(x_i) = q''(x_i) = f''(x_i).$$

Let $r = p - q$. So, $r$ is a polynomial of degree (at most) $2n+1$. Thus $r''$ is a polynomial of degree (at most) $2n-1$. We would like to claim it has $2n$ zeros, in which case it would have to be identically zero. Our claim would be based on these facts:

(i) $r$ has $n+1$ zeros, since $r(x_i) = p(x_i) - q(x_i) = f(x_i) - f(x_i) = 0$, for $i = 0, 1, \ldots, n$. By Rolle's theorem, $r'(x)$ has $n$ zeros, which are located in $(x_0, x_1)$, $(x_1, x_2)$, ..., $(x_{n-1}, x_n)$. Again applying Rolle's Theorem, gives that $r''$ has $n-1$ zeros.

(ii) Since $p''(x_i) = q''(x_i) = f''(x_i)$, we get that $r$ has a further $n+1$ zeros.

However, it is not possible to deduce that the zeros coming from (i) are different from those coming from (ii). So we cannot say that $r''$ must have $2n$ zeros.

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(ii) *Show that there is no $p_5(x)$ that solves this problem when*

- $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.
- $f(-1) = 1$, $f(0) = 0$, $f(1) = 1$.
- $f''(-1) = 0$, $f''(0) = 0$, $f''(1) = 0$.

**Solution:** Write $p_5 = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5$. Then $f(0) = 0$ gives that $c_0 = 0$.

Also, $p_5'' = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3$. Then $f''(0) = 0$ gives that $c_2 = 0$.

Next $f(-1) = 1$ gives that $-c_1 - c_3 + c_4 - c_5 = 1$.
And $f(1) = 1$ gives that $c_1 + c_3 + c_4 + c_5 = 1$.
Adding these gives that $c_4 = 1$.

Similarly, $f''(-1) = 0$ gives that $-6c_3 + 12c_4 - 20c_5 = 0$.
And $f''(1) = 0$ gives that $6c_3 + 12c_4 + 20c_5 = 0$.
Adding these gives that $c_4 = 0$.

Since we can't have that $c_4 = 0$ and $c_4 = 1$ we can't satisfy all the equations: there is no solution.