Chapter 1

Polynomial Interpolation

1.1 Introduction

Suppose that we have a two sets of $n+1$ real numbers $\{x_i\}_{i=0}^{n+1}$ and $\{y_i\}_{i=0}^{n+1}$, and that the x_i are *strictly* increasing: $x_0 < x_1 < x_2 < \cdots < x_n$. *Interpolation* problems are of the form: *Find a function* p*, that is continuous and defined on* $[x_0, x_n]$*, such that*

$$p(x_k) = y_k, \quad \textit{for} \quad k = 0, 1, \ldots, n.$$

We say that p *interpolates* the points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n).

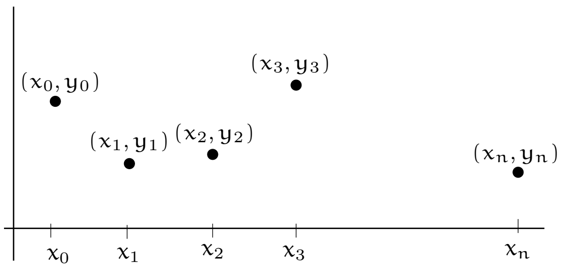

Fig. 1.1: Some points to interpolate

Why Bother? There are several possibilities, including

- The points belong to an underlying, but unknown function. We wish to establish likely values of f at points other than x_0, x_1, ..., x_n. The values of f may have been obtained from physical experiments, or numerical procedures (e.g., Newton's method for initial value problems). Or it may be that some values of the function are easily available. For example $2! = 2$, and $3! = 6$, but what about $2\frac{1}{2}!$ or $\pi!$?
- We may know the function, but prefer to work with an interpolant to it. For example, in order to estimate derivatives or integrals of a function.

Applications? In mathematics, from number theory to information theory, and nearly every aspect of numerical analysis. Elsewhere, the methods are used in fields ranging from aircraft design to computer animation.

The main reference for this section is Chapter 6 of [SM03]. See also, Lectures 18–20 of [S96].

1.1.1 Polynomial Interpolation

Definition 1.1.1. $\mathcal{P}_n$ is the set of polynomials of degree less than or equal to n and real-valued coefficients, i.e., $p_n \in \mathcal{P}_n$ if

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $a_i \in \mathbb{R}$.

Examples:

`Take notes:`

It is particularly important to note that if p_n and q_n both belong to $\mathcal{P}_n$, then so too does their sum.

The Polynomial Interpolation Problem comes in two forms.

> **Polynomial Interpolation Problem 1 (PIP1)**: *Given is set of points* $x_0 < x_1 < \cdots < x_n$*, and a set of real numbers* $y_0, y_1, \ldots, y_n$*, find* $p_n \in \mathcal{P}_n$ *such that*
> $$p_n(x_k) = y_k, \quad \textit{for } k = 0, 1, \ldots, n. \tag{1.1}$$

> **Polynomial Interpolation Problem 2 (PIP2)**: *Given is set of points* $x_0 < x_1 < \cdots < x_n$*, and a function* $f : [x_0, x_n] \to \mathbb{R}$*, find* $p_n \in \mathcal{P}_n$ *such that*
> $$p_n(x_k) = f(x_k), \quad \textit{for } k = 0, 1, \ldots, n. \tag{1.2}$$

Clearly PIP2 is just PIP1 with $y_k = f(x_k)$.

The questions that we must ask (and answer) are

(i) Is there a solution to the polynomial interpolation problem.

(ii) Is it unique?

(iii) How do we find it?

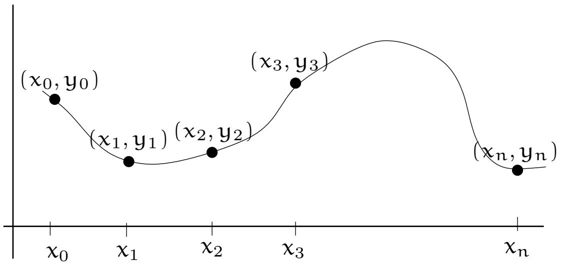

Fig. 1.2: A possible interpolation of the points in Figure 1.1

(iv) How accurate is it? That is, if f is the underlying function (i.e., $f(x_k) = y_k$), can we find an upper bound for

$$\max_{x_0 \leqslant x \leqslant x_n} \{|f(x) - p_n(x)|\}?$$

1.1.2 Exercises

Exercise 1.1. Suppose that $p \in \mathcal{P}_m$ and $q \in \mathcal{P}_n$.

(a) What is the maximum possible degree of $p + q$?

(b) What is the minimum possible degree of $p - q$?

(c) What is the maximum possible degree of pq?

Exercise 1.2. Find out what a *vector space* is. Convince yourself that $\mathcal{P}_n$ is a vector space.

Exercise 1.3. (a) Is it always possible to find a polynomial of degree 1 that interpolates the single point (x_0, y_0)? If so, how many such polynomials are there? Explain your answer.

(b) Is it always possible to find a polynomial of degree 1 that interpolates the two points (x_0, y_0) and (x_1, y_1)? If so, how many such polynomials are there? Explain your answer.

(c) Is it ever possible to find a polynomial of degree 1 that interpolates the three points (x_0, y_0), (x_1, y_1), and (x_2, y_2)? If so, give an example.