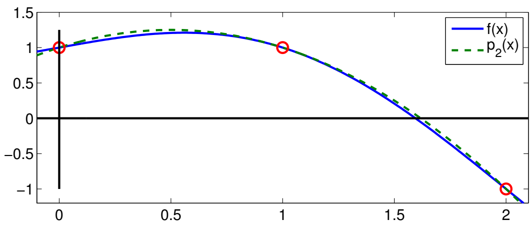
1.2 Finding the polynomial

1.2.1 An example

Example 1.2.1. Show that the polynomial of degree 2 that interpolates $f(x) = 1 - x + \sin(\pi x/2)$ at the points $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$ is $p_2 = -x^2 + x + 1$.



Take notes:

1.2.2 Uniqueness

It is not hard to convince ourselves that $-x^2 + x + 1$ is the solution to the PIP in Example 1.2.1. But how to we know we have found the *only* solution? More generally, *under what conditions is there exactly one polynomial that solves the PIP*?

Lemma 1.2.2. *If $p_n \in \mathcal{P}_n$ has $n+1$ zeros, then $p_n \equiv 0$ (i.e., $p_n(x) = 0$ for all x).*

Take notes:

Theorem 1.2.3 (There is a unique solution to the PIP). *There is at most one polynomial of degree at most n that interpolates the $n+1$ points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) where x_0, x_1, ..., x_n are distinct.*

Take notes:

1.2.3 The Vandermonde matrix method

In Example 1.2.1 we gave a polynomial the solves a particular PIP, but this didn't tell us *how* to find the solution. It turns out that the most obvious approach may not be the best.

Suppose we are trying to solve the problem as follows: *find p_2 such that*

$$p_2(x_0) = y_0, \quad p_2(x_1) = y_1, \quad \textit{and} \quad p_2(x_2) = y_2.$$

Since $p_2(x)$ is of the form $a_0 + a_1x + a_2x^2$, this just amounts the finding the values of the coefficients a_0, a_1, and a_2. One might be tempted to solve for them using the system of equations

$$\begin{aligned} a_0 + a_1x_0 + a_2x_0^2 &= y_0 \\ a_0 + a_1x_1 + a_2x_1^2 &= y_1 \\ a_0 + a_1x_2 + a_2x_2^2 &= y_2. \end{aligned}$$

This is known as the *Vandermonde System*. [1]. Writing it in matrix-vector format we get:

$$\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}, \quad \text{or} \quad V\mathbf{a} = \mathbf{y}. \tag{1.3}$$

But this may not be a good idea. (*You can skip the next bit if you did not take MA385*).

In MA385 we learned about the relationship between the *condition number* of a matrix, V, and the relative error in the (numerical) solution to a matrix-vector equation with V as the coefficient matrix. The condition number is

$$\kappa(V) = \|V\|\|V^{-1}\|,$$

for some subordinate matrix norm $\|\cdot\|$.

Example 1.2.4. ([S96, Lecture 18]) Suppose $x_0 = 100$, $x_1 = 101$ and $x_2 = 102$. Then it is not hard to check that

$$\|X\|_\infty = \max_i \sum_j |X_{ij}| = 10,507.$$

Also,

$$V^{-1} = \frac{1}{2}\begin{pmatrix} 10302 & -20400 & 10100 \\ -203 & 404 & -201 \\ 1 & -2 & 1 \end{pmatrix},$$

so $\|V^{-1}\|_\infty = 20401$. So $\kappa(V) = 214,353,307$.

[1] *Alexandre-Théophile Vandermonde* (France, 1735–1796) only began studying mathematics at the age of 35. Prior to that, he was a professional musician. Is credited for inventing the Vandermonde Matrix (and determinant), which he didn't do, but not for publishing the first paper in modern algebra, which he may have done.

1.2.4 Lagrange Interpolation

We'll now look at a much easier method for solving the Polynomial Interpolation Problem. As a by-product, we get a constructive proof of the existence of a solution to the PIP. (Here "constructive" means that we'll prove it exists by actually computing it).

Example 1.2.5. Consider the problem: *find* $p_3 \in \mathcal{P}_3$ *such that*

$$p_3(0) = 2, \quad p_3(1) = -1/2, \quad p_3(2) = 1, \quad p_3(3) = -1.$$

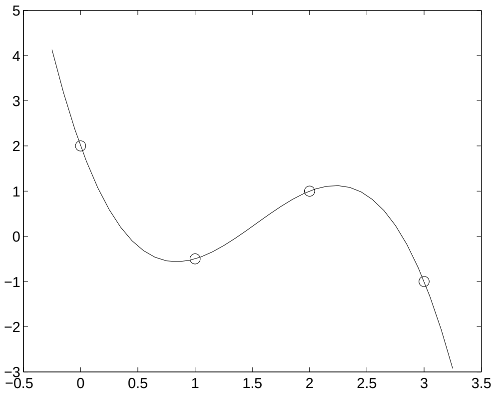


Fig. 1.1: A possible solution to Example 1.2.5

This is not so easy to solve; the following problem is easier: *find* $L_0 \in \mathcal{P}_3$ *such that*

$$L_0(0) = 1, \quad L_0(1) = 0, \quad L_0(2) = 0, \quad L_0(3) = 0.$$

Obviously, because L_0 is a cubic and has zeros at $x = 1, 2, 3$ it is of the form $L_0(x) = C(x-1)(x-2)(x-3)$. Then, choosing C so that $L_0(0) = 1$, we get

$$L_0(x) =$$

Similarly, let $L_1 \in \mathcal{P}_3$ be the cubic polynomial such that

$$L_1(0) = 0, \quad L_1(1) = 1, \quad L_1(2) = 0, \quad L_1(3) = 0,$$

then

$$L_1(x) =$$

In the same style, $L_2(x_i) = \begin{cases} 1 & i = 2 \\ 0 & i = 0, 1, 3 \end{cases}$ gives

$$L_2(x) =$$

Also, if we define

$$L_3(x_i) = \begin{cases} 1 & i = 3 \\ 0 & i = 0, 1, 2 \end{cases},$$

then clearly,

$$L_3(x) = \frac{(x-0)(x-1)(x-3)}{(3-0)(3-1)(3-2)} = \prod_{j=0, j \neq 3}^{n} \frac{(x - x_j)}{(x_3 - x_j)}.$$

Plots of these polynomials are shown in Figure 1.2.

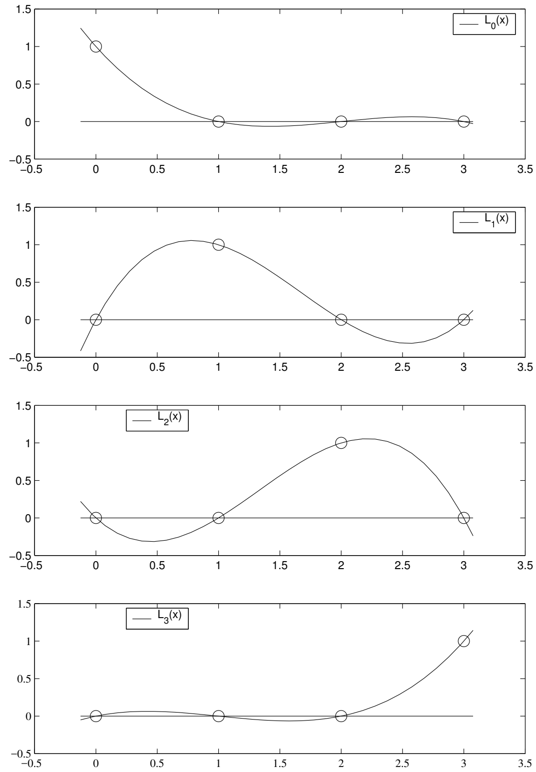


Fig. 1.2: The polynomials $L_0(x)$, $L_1(x)$, $L_2(x)$, and $L_3(x)$

Because each of L_0, L_1, L_2, and L_3 is a cubic, so too is any linear combination of them. So

$$p_3(x) = 2L_0(x) - (1/2)L_1(x) + (1)L_2(x) + (-1)L_3(x),$$

is a cubic. Furthermore

$$\begin{array}{lcllll}
p_3(0) & = & 2L_0(0) & -(1/2)L_1(0) & +(1)L_2(0) & +(-1)L_3(0) \\
 & = & 2(1) & -(1/2)(0) & +(1)(0) & +(-1)(0) \\
 & = & 2, & & & \\
p_3(1) & = & 2L_0(1) & -(1/2)L_1(1) & +(1)L_2(1) & +(-1)L_3(1) \\
 & = & 2(0) & -(1/2)(1) & +(1)(0) & +(-1)(0) \\
 & = & -1/2, & & & \\
p_3(2) & = & 2L_0(2) & -(1/2)L_1(2) & +(1)L_2(2) & +(-1)L_3(2) \\
 & = & 2(0) & -(1/2)(0) & +(1)(1) & +(-1)(0) \\
 & = & 1, & & & \\
p_3(3) & = & 2L_0(3) & -(1/2)L_1(3) & +(1)L_2(3) & +(-1)L_3(3) \\
 & = & 2(0) & -(1/2)(0) & +(1)(0) & +(-1)(1) \\
 & = & -1. & & &
\end{array}$$

Thus p_3 solves the problem in Example 1.2.5

1.2.5 The Lagrange Form

We can generalise this idea to solve any PIP using what is called *Lagrange*[2] interpolation.[3] We'll now look how to solve the general problem.

Definition 1.2.6. The **Lagrange Polynomials** associated with $x_0 < x_1 < \cdots < x_n$ is the set $\{L_i\}_{i=0}^{n}$ of polynomials in $\mathcal{P}_n$ such that

$$L_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \tag{1.4a}$$

and are given by the formula

$$L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}. \tag{1.4b}$$

Definition 1.2.7. The **Lagrange form of the Interpolating Polynomial**[a]

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x), \tag{1.5a}$$

or

$$p_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x). \tag{1.5b}$$

[a]Take care not to confuse the *Lagrange Polynomials*, which are the L_i with the *Lagrange Interpolating Polynomial*, which is the p_n defined in (1.5).

Theorem 1.2.8 (Lagrange)**.** *There exists a solution to the Polynomial Interpolation Problem and it is given by*

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x).$$

<u>`Take notes`</u>:

Example 1.2.9. [SM03, E.g. 6.1] Write down the Lagrange form of the polynomial interpolant to the function $f(x) = e^x$ at interpolation points $\{-1, 0, 1\}$.

<u>`Take notes`</u>:

Figure 1.3 shows the solution to Example 1.2.9 (top) and the difference between the function e^x and its interpolant (bottom). It would be interesting to see how this error depends on

(i) the function (and its derivatives)

(ii) the number of points used.

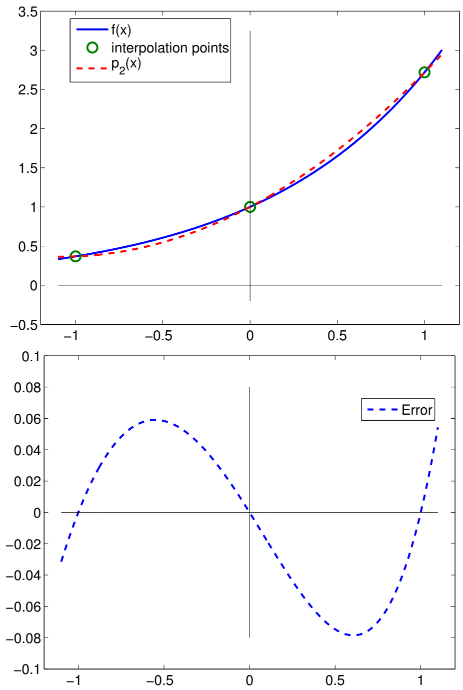


Fig. 1.3: The solution to Eg. 1.2.9 and the error

1.2.6 Some terminology

- The process of finding a polynomial, p_n, that solves the PIP is called *interpolation*.
- The solution, p_n, is called an *interpolant*.
- The difference between $f(x)$ and $p_n(x)$ is called the *interpolation error*.

2 

Joseph-Louis Lagrange, born 1736 in Turin, died 1813 in Paris. He made great contributions to many areas of Mathematics, including *Calculus of Variations*, which we'll see a little of at the end of the semester. Image source: `http://jeff560.tripod.com/stamps.html`

[3]This formula was actually discovered by E. Waring in 1776; rediscovered by Euler in 1783, and published by Lagrange in 1795

1.2.7 Exercises

Exercise 1.4. Give a proof of Lemma 1.2.2 that is different from the one we did in class.

Exercise 1.5. The general form of the *Vandermonde* Matrix is

$$V_n = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

Its determinant is

$$\det(V_n) = \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i). \tag{1.6}$$

Verify this for the 2×2 and 3×3 cases.

(Note that from formula (1.6) we can deduce directly that the PIP has a unique solution *if and only if* the points $x_0, x_1, \dots, x_n$ are all distinct.)

Aside: Here is a hint for proving that in general

$$\det(V_n) = \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i).$$

First note that $\det(V_n) = \det(V_n^T)$ and now consider the determinant of

$$V_n^T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_0 & x_1 & x_2 & \cdots & x_n \\ x_0^2 & x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_0^n & x_1^n & x_2^n & \cdots & x_n^n \end{pmatrix}.$$

Using elementary row operations (which preserve the determinant), add to each row, the row above it multiplied by $-x_0$, to show that we need to compute the determinant of

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_1 - x_0 & x_2 - x_0 & \cdots & x_n - x_0 \\ 0 & x_1^2 - x_1 x_0 & x_2^2 - x_2 x_0 & \cdots & x_n^2 - x_n x_0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & x_1^n - x_1^{n-1} x_0 & x_2^n - x_2^{n-1} x_0 & \cdots & x_n^n - x_n^{n-1} x_0 \end{pmatrix}.$$

Now try to continue by induction...

Exercise 1.6. ⋆ (MA378 Semester 2 exam, 2015/2016) Let p_3 be the polynomial that interpolates $f(x) = \sin(x/2)$ at the points $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$ and $\mathbf{x_3 = 1}$. Use Cauchy's Theorem to give a bound for the error at $x = 1/2$. (You don't have to give a formula for p_3).[4].

Exercise 1.7. Find the polynomial p_1 that interpolates the function $f(x) = x^3$ at the points $x_0 = 0$ and $x_1 = a$. Find the point $\sigma \in [0, a]$ that maximises $|f(x) - p_1(x)|$, and hence compute $\max_{0 \leqslant x \leqslant a} |f(x) - p_1(x)|$.
Source: Chapter 6 of Süli and Mayers.

[4]An earlier version of this stated, incorrectly, that $x_4 = 1$

Exercise 1.8. ⋆ Show that

$$\sum_{i=0}^{n} L_i(x) = 1 \quad \text{for all } x.$$

Exercise 1.9. Write down the Lagrange Form of p_2, the polynomial of degree 2 that interpolates the points $(0, 3)$, $(1, 2)$ and $(2, 4)$.
Source: Chapter 2 of Stoer and Bulirsch.

Exercise 1.10. Show that all the following represent the same polynomial (usually called the "Chebyshev Polynomial of Degree 3"), $T_3(x) = 4x^3 - 3x$.

(a) Horner form: $\big((4x + 0)x - 3\big)x + 0$.

(b) Lagrange form: $\displaystyle\sum_{k=0}^{3} \Big(\prod_{j=0, j \neq k}^{3} \frac{x - x_j}{x_k - x_j} \Big) (-1)^{k+1}$, where $x_0 = -1, x_1 = -1/2, x_2 = 1/2, x_3 = 1$.

(c) Recurrence relation: $T_0 = 1$, $T_1 = x$, and $T_n = 2xT_{n-1} - T_{n-2}$ for $n = 2, 3, \ldots$

(d) Trigonometric form: $T_3(x) = \cos\big(3\cos^{-1}(x)\big)$.