1.3 Polynomial Interpolation Errors

1.3.1 Introduction

In Example 1.2.9, we wrote down the polynomial of degree $n = 2$ interpolating $f(x) = e^x$ at $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$.

We now want to investigate how, in general, error in polynomial interpolation depends on

(i) the function (and its derivatives)

(ii) the number of points used (or, equivalently, degree of the polynomial used).

The main ingredient we need is the following theorem.

Theorem 1.3.1 (Rolle's[5] Theorem)**.** *Let g be a function that is continuous and differentiable on the interval $[a, b]$. If $g(a) = g(b)$, then there is at least one point c in (a, b) where $g'(c) = 0$.*

Our "proof" is by picture:[6]
`Take notes`:

1.3.2 Error estimate for n=0

The simplest case is when $n = 0$, so the interpolant is a constant, i.e., it is p_0 interpolating a function f at a point x_0. Here is one way we can deduce the *interpolation error*.[7]
`Take notes`:

It is important to understand what this formula is telling us:
`Take notes`:

1.3.3 Error estimates for $n \geqslant 1$

We will use this Rolle's Theorem to prove the most important theorem of NA2; it is used repeatedly through-out the course. It's often called the *Polynomial Interpolation Error Theorem*, but we'll call it *Cauchy's Theorem* for short.

First, we need to define an important polynomial.

Definition 1.3.2. The **Nodal Polynomial** π_{n+1} associated with the interpolation points that $a = x_0 < x_1 < \cdots < x_n = b$ is

$$\pi_{n+1}(x) = (x - x_0)(x - x_1)\dots(x - x_n) = \prod_{i=0}^{n}(x - x_i).$$

Theorem 1.3.3 (Cauchy, 1840)**.** *Suppose that $n \geqslant 0$ and f is a real-valued function that is continuous and defined on $[a, b]$, such that the derivative of f of order $n + 1$ exists and is continuous on $[a, b]$. Let p_n be the polynomial of degree n that interpolates f at the $n + 1$ points $a = x_0 < x_1 < \cdots < x_n = b$. Then, for any $x \in [a, b]$ there is a $\tau \in (a, b)$ such that*

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\pi_{n+1}(x). \qquad (1.7)$$

Proof: `Take notes`:

[5] Michel Rolle, Born 1652 in Ambert, Basse-Auvergne (France), died 1719 in Paris. This is his most famous result.

[6] One can easily deduce Rolle's Theorem from the Mean Value Theorem (MVT). But since the standard proof of the MVT uses Rolle's Theorem, that would be cheating

[7] There is an easier way, involving the MVT, but the approach given here is easier to generalise

Example 1.3.4. Recall Theorem 1.2.9, where we wrote down the Lagrange form of the polynomial, p_2, that interpolates $f(x) = e^x$ at the points $\{-1, 0, 1\}$. Give a formula for $e^x - p_2(x)$.

Take notes:

Usually (and as in the above example), we can't calculate $f(x) - p_n(x)$ exactly from Formula (1.7), because we have no way of finding τ. However, we are typically not so interested in what the error is at some given point, but what is the maximum error over the whole interval $[x_0, x_n]$. That is given by:

Corollary 1.3.5. *Define*

$$M_{n+1} = \max_{x_0 \leqslant \sigma \leqslant x_n} |f^{(n+1)}(\sigma)|.$$

Then

$$|f(x) - p_n(x)| \leqslant \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)|. \tag{1.8}$$

Example 1.3.6. Let p_1 be the polynomial of degree 1 that interpolates a function f at distinct points x_0 and x_1. Letting $h = x_1 - x_0$, show that

$$\max_{x_0 \leqslant x \leqslant x_1} |f(x) - p_1(x)| \leqslant \frac{1}{8} h^2 M_2.$$

Take notes:

1.3.4 Exercise

Exercise 1.11. Let p_2 be the polynomial of degree 2 that interpolates a function f at the points x_0, x_1 and x_2. If $x_1 - x_0 = x_2 - x_1 = h$, show that

$$\max_{x_0 \leqslant x \leqslant x_2} |f(x) - p_2(x)| \leqslant \frac{1}{6} \frac{2}{3\sqrt{3}} h^3 M_3 = \frac{1}{9\sqrt{3}} h^3 M_3.$$

Hint: simplify the calculations by taking $t = x - x_1$, *writing* $(x - x_0)(x - x_1)(x - x_2)$ *in terms of* h *and* t.