1.5 Hermite Interpolation

1.5.1 The idea

Hermite[9] interpolation is a variant on the standard Polynomial Interpolation Problem: we seek a polynomial that not only agrees with a given function f at the interpolation points, but its first derivative also matches f' at those points. We are not that interested in this problem for its own sake, but the idea recurs again in the sections in piecewise polynomial interpolation and Gaussian quadrature.

Formally, the problem is

The Hermite Polynomial Interpolation Problem (HPIP)

Given a set of interpolation points $x_0 < x_1 < \cdots < x_n$ and a continuous, differentiable function f, find $p_{2n+1} \in \mathcal{P}_{2n+1}$ such that

$$p_{2n+1}(x_i) = f(x_i) \quad \textit{and} \quad p'_{2n+1}(x_i) = f'(x_i).$$

It is not hard to see that if there is a solution to this problem, then it is unique:

Take notes:

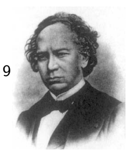
1.5.2 Constructing Hermite Interpolants

It is possible to solve this problem using an extension of the Lagrange Polynomial approach of §1.2.4. Given the Lagrange Polynomials L_i as defined in (1.4) let

$$H_i(x) = [L_i(x)]^2(1 - 2L_i'(x_i)(x - x_i)),$$

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

Examples of two such functions are shown in Figure 1.1.



9 Charles Hermite, France, 1822–1901. A part from this form of interpolation, his contributions to Mathematics included the first proof that e is transcendental. His methods were later used to show that π is transcendental.

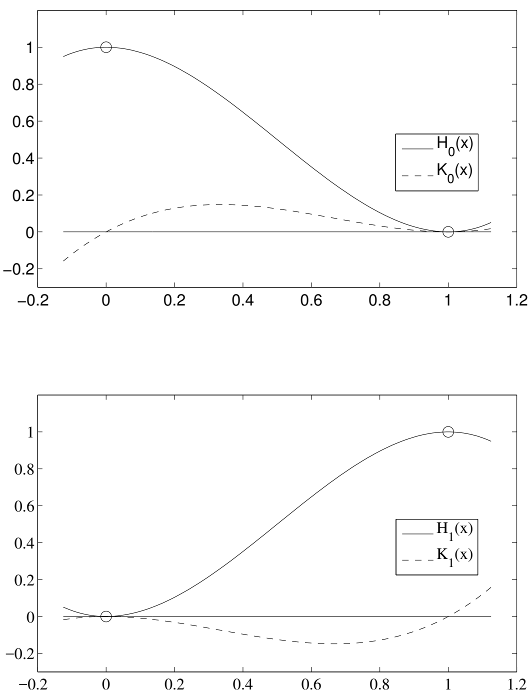


Fig. 1.1: Hermite bases functions H_0, K_0 (top) and H_1, K_1 (bottom) for $n = 1$, $x_0 = 0$ and $x_1 = 1$

We can show that, for $i, k = 0, 1, \ldots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad H_i'(x_k) = 0 \ \forall k,$$

$$K_i(x_k) = 0, \qquad K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

We'll show this for H and H', and leave the corresponding results for K and K' to Exercise 1.16.

Take notes:

Armed with these identities, we can now show that the solution to the HPIP exists and is given by

$$p_{2n+1}(x) = \sum_{i=0}^{n} \big(f(x_i)H_i(x) + f'(x_i)K_i(x)\big).$$

(Again, see Exercise 1.16 for details).

Example 1.5.1. Find the polynomial of degree 3 that interpolates $\exp(x^2)$, and its first derivative, at $x_0 = 0$ and $x_1 = 1$. (See Figure 1.2).

`Take notes`:

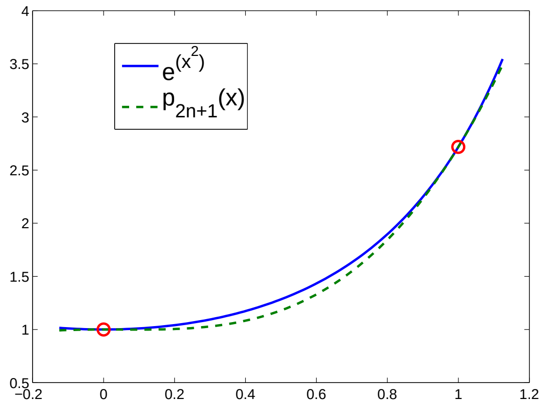


Fig. 1.2: The Hermite interpolant to $e^{(x^2)}$ at $x = 0$ and $x = 1$

Theorem 1.5.2. *Let be a real-valued function that is continuous and defined on* $[a, b]$*, such that the derivatives of* f *of order* $2n + 2$ *exist and are continuous on* $[a, b]$*. Let* p_{2n+1} *be the Hermite interpolant to* f*. Then, for any* $x \in [a, b]$ *there is an* $\tau \in (a, b)$ *such that*

$$f(x) - p_{2n+1}(x) = \frac{f^{(2n+2)}(\tau)}{(2n+2)!}[\pi_{n+1}(x)]^2.$$

We won't do a proof of this in class. However, later in this course we'll be interested in the particular example of finding p_3 the cubic Hermite Polynomial Interpolant to a function f at the points x_0 and x_1. Also, see Exercise 1.13.

1.5.3 Exercises

Exercise 1.13. $\star$ For *just* the case $n = 1$, state and prove an appropriate version of Theorem 1.5.2 (i.e., error in the Hermite interpolant). Use this to find a bound for $\|f - p_3\|_{[x_0,x_1]}$ in terms of f and $h = x_1 - x_0$. (Here $\|g\|_{[x_0,x_1]}$ is short-hand for $\max_{x_0 \leqslant x \leqslant x_1} |g(x)|$.)

Exercise 1.14. Let $n = 2$ and $x_0 = -1$, $x_0 = 1$ and $x_1 = 1$. Write out the formulae for H_i, K_i for $i = 0, 1, 2$ and give a rough sketch of each of these six functions that shows the value of the function and its derivative at the three interpolation points.

Exercise 1.15. Do Exercise 6.6 from from Süli and Mayers, *An Introduction to Numerical Analysis*.

Exercise 1.16. $\star$ Let L_0, L_1, ..., L_n be the usual Lagrange polynomials for the set of interpolation points $\{x_0, x_1, \dots, x_n\}$. Now define

$$H_i(x) = [L_i(x)]^2(1 - 2L_i'(x_i)(x - x_i)),$$

and

$$K_i(x) = [L_i(x)]^2(x - x_i).$$

We saw in class that, for $i, k = 0, 1, \dots n$,

$$H_i(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad H_i'(x_k) = 0.$$

Show that, for $i, k = 0, 1, \dots n$,

$$K_i(x_k) = 0, \qquad K_i'(x_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

Conclude that the solution to the Hermite Polynomial Interpolation Problem is

$$p_{2n+1}(x) = \sum_{i=0}^{n} \big(f(x_i)H_i(x) + f'(x_i)K_i(x)\big).$$

Exercise 1.17. Write down that formula for q_3, the *Hermite* polynomial that interpolates $f(x) = \sin(x/2)$, and its derivative, at the points $x_0 = 0$ and $x_1 = 1$. Give an upper bound for $|f(1/2) - q_3(1/2)|$. How does this compare with the bound in Exercise 1.6?

Exercise 1.18. (This exercise is based on Exer 6.5 from Süli and Mayers' *Introduction to Numerical Analysis*). Consider the following problem.

> Take $n+1$ distinct interpolation points $x_0 < x_1 < \dots < x_n$. Let p_{2n+1} be the polynomial of degree $2n + 1$ with the property that
>
> $$p_{2n+1}(x_i) = f(x_i),$$
>
> and
>
> $$p''_{2n+1}(x_i) = f''(x_i).$$

In general this problem does *not* have a unique problem.

(i) Explain briefly but carefully why the arguments, based on Rolle's Theorem, used to prove **uniqueness** of solutions to the HPIP, will not work here.

(ii) Show that there is no $p_5(x)$ that solves this problem when

- $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.
- $f(-1) = 1$, $f(0) = 0$, $f(1) = 1$.
- $f''(-1) = 0$, $f''(0) = 0$, $f''(1) = 0$.