1.6 Wrap-up

1.6.1 Convergence & Runge's Example

The celebrated Weierstrass approximation theorem states that, given f and a positive number ε, there is a polynomial p such that

$$\max_{x\in[a,b]} |f(x) - p(x)| := \|f - p\|_\infty \leqslant \varepsilon.$$

Now suppose that f is a continuous function on $[a, b]$ and that $\{p_n\}_{n=0}^{\infty}$ is a sequence of polynomials that interpolate f at $n + 1$ **equally spaced** points. One might be inclined to believe that

$$\lim_{n\to\infty} \|f - p_n\|_\infty = 0.$$

Equivalently, recalling (1.8):

$$|f(x) - p_n(x)| \leqslant \frac{M_{n+1}}{(n+1)!}|\pi_{n+1}(x)|,$$

one might expect that

$$\lim_{n\to\infty} \max_{x\in[a,b]} \frac{M_{n+1}}{(n+1)!}|\pi_{n+1}(x)| = 0.$$

In order words, we might think that, in order to find an interpolating polynomial that is as accurate as we would like, we just need to choose large enough n.

And some times it might work. For example, suppose that $a = -5$, $b = 5$, and $f(x) = e^{\sin(x/2)}$. In Table 1.1 the errors for successive interpolants are shown. A few of these are shown in Figure 1.3.

Table 1.1: Errors in polynomial interpolants to $e^{\sin(x/2)}$ on $[-5, 5]$

n	$\|f - p_n\|_\infty$
2	1.27e-00
4	2.94e-01
6	8.39e-02
8	5.75e-02
16	1.07e-03

However, there is a famous example of a simple function that cannot be successfully interpolated in this manner. This is *Runge's Example*:

$$f(x) = \frac{1}{1+x^2} \qquad \text{on } [-5, 5].$$

The errors are shown in Table 1.2. Notice that they *increase* with n. We don't have the scope to go into *why* this is the case (but see the notes from class on a heuristic explanation).

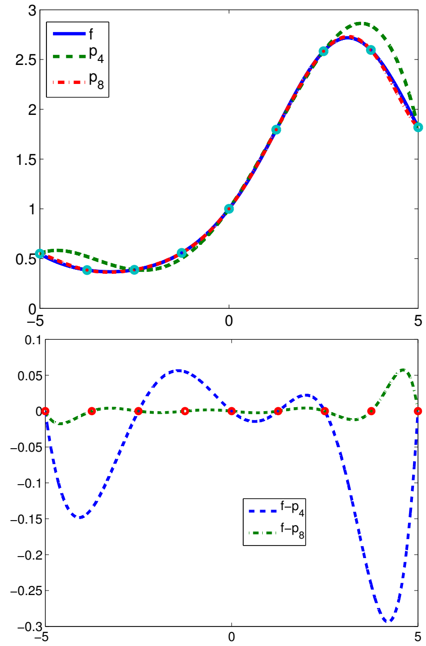

Fig. 1.3: Polynomial interpolants to $e^{\sin(x/2)}$ on $[-5, 5]$, and their errors (bottom)

Table 1.2: Errors in interpolants to $1/(1 + x^2)$ on $[-5, 5]$

n	$\|f - p_n\|$
2	0.65
4	0.44
6	0.62
8	1.05
16	14.39
20	59.66
22	122.91
24	257.21

1.7 Where to from here?

So now it looks like polynomial interpolation is bad, at least on equidistant points. However, our experience in Lab 1 might lead us to be more optimistic: we were able to find a set of points that made the approximation as good we wanted (until round-off error dominated).

Unfortunately, just because we have a good set of points for interpolating one particular function, it does not follow that that set is good for every continuous function: this is *Faber's Theorem*[10]. This has often led numerical analysts to abandon the idea of interpolation by high-order polynomials completely.

However, there is a set of points that are useful, if f is smooth enough: the Chebyshev points of Lab 1. If you are interested, there please look at the essay *Inverse Yogiisms* by Lloyd N. (Nick) Trefethen, Notices AMS, Dec. 2016.[11] To investigate this numerically in MATLAB, try

[10] Georg Faber (18771966)

[11] http://people.maths.ox.ac.uk/trefethen/essays.html or http://www.ams.org/publications/journals/notices/

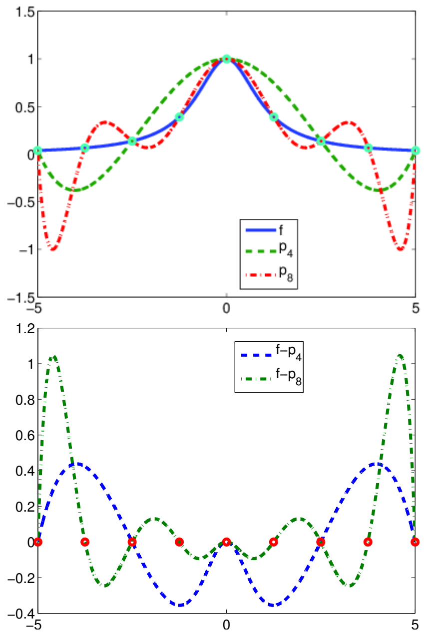

Fig. 1.4: Polynomial interpolants to $1/(1+x^2)$ on $[-5, 5]$

exploring the Chebfun toolbox.

The approach we will take is different. In (1.3.6) we saw that

$$\max_{x_0 \leqslant x \leqslant x_1} |f(x) - p_1(x)| \leqslant \frac{1}{8} h^2 M_2.$$

So, assuming M_2 is bounded (which is reasonable), we can make p_1 as close to f as we would like by taking a small enough interval $[x_0, x_1]$. The next section of this module is devoted to seeing how this can be used in theory and practice.