2.2 Cubic Splines

The Cubic Spline[2] is perhaps the most useful and popular interpolating function used in numerical analysis, automotive and aeronautical engineering, computer and film animation, digital photography, financial modelling, and as many other areas as there are human endeavours where continuous processes are modelled by discrete ones.

One could argue that the most fundamental shortcoming of the linear spline interpolant to f (see, for example, Figure 2.1) is that they are not "smooth": that is, although $l_i(x_i) = l_{i+1}(x_i)$, it is **not** generally the case that $l_i'(x_i) = l_{i+1}'(x_i)$. Also, the interpolating function cannot capture the "curvature" of f. (If you think about this last statement, you'll see that it can be expressed as $l''(x) = 0$ for all $x \in [a, b]$.)

Cubic splines try to balance the simplicity of the linear spline approach — by using a low-order polynomial on each interval — with the desire for curvature and more smoothness – by using cubics, and by forcing adjacent ones, and their first and second derivatives, to agree at knot points.

Definition 2.2.1. Let f be a function that is continuous on $[a, b]$. The *cubic spline interpolant* to f is the continuous function S such that

(i) for $i = 1, \ldots, N$, on each interval $[x_{i-1}, x_i]$ let $S(x) = s_i(x)$, where each of the s_i is a cubic polynomial.

(ii) $s_i(x_{i-1}) = f(x_{i-1})$ for $i = 1, \ldots, N$,

(iii) $s_i(x_i) = f(x_i)$ for $i = 1, \ldots, N$,

(iv) $s_i'(x_i) = s_{i+1}'(x_i)$ for $i = 1, \ldots, N-1$,

(v) $s_i''(x_i) = s_{i+1}''(x_i)$ for $i = 1, \ldots, N-1$.

So, we have defined the cubic spline S as a function that interpolates f at $N+1$ points, has continuous first and second derivatives on $[x_0, x_N]$ and is a cubic polynomial on each of the n intervals $[x_{i-1}, x_i]$. That is, it is *piecewise cubic*. We know that one can write a cubic as

$$p_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Thus it takes 4 terms to uniquely define a single cubic. To uniquely determine N cubics requires $4N$ terms. They can be found by solving $4N$ (linearly independent) equations. But an inspection of (ii)–(iv) above reveals only $4N - 2$ equations.

[2] It is generally accepted that (mathematical) splines were first described by Isaac Schoenberg (1903-1990) during WW2. However their physical realisation had been used in the ship-building and aircraft industries prior to that.

The "missing" equations can be chosen in a number of ways:

(i) by setting $S''(x_0) = 0$ and $S''(x_N) = 0$. This is called a *natural* spline, and is the approach we'll take.

(ii) by setting $S'(x_0) = 0$ and $S'(x_N) = 0$. This is called a *clamped* spline.

(iii) set $S'(x_0) = S'(x_N)$ and $S''(x_0) = S''(x_N)$. This is the *periodic* spline and is used for interpolating, say, trigonometric functions.

(iv) only use $N-2$ components of the spline: $s_2, \ldots, s_{N-1}$. But extend to two end ones so that $s_2(x_0) = f(x_0)$ and $s_{N-1} = f(x_N)$. This is called the *not-a-knot* condition.

In Figure 2.3 we show the natural cubic spline interpolants to $f(x) = 1/(1+x^2)$ (with $a = -5$ and $b = 5$ as usual) on $N = 4$, 8 and 16 intervals. Compared with Figure 2.1, we seem to get significantly better approximation. Moreover, the rate at which the error decreases with respect to h is much faster.

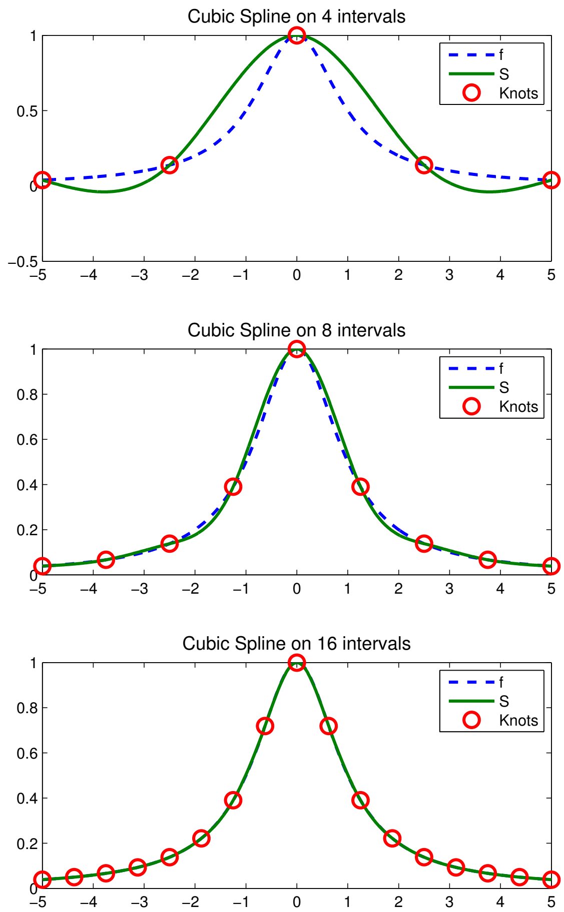

Fig. 2.3: Cubic spline interpolants to $1/(1+x^2)$. Compare with Figure 2.1

2.2.1 Constructing Cubic Splines

Some notation:

- $h = x_i - x_{i-1} = (x_N - x_0)/N$ for all i.
- $f_i := f(x_i)$.

To construct the spline, first observe that if S is piecewise cubic, then S' is piecewise quadratic, and S'' is piecewise linear.

(i) Let $\sigma_i = S''(x_i)$ for $i = 0, \ldots, N$.

Take notes:

(ii) Integrate twice:

Take notes:

We get

$$s_i(x) = \alpha_i(x - x_{i-1}) + \beta_i(x_i - x) + \frac{\sigma_{i-1}}{6h}(x_i - x)^3 + \frac{\sigma_i}{6h}(x - x_{i-1})^3, \quad (2.2)$$

for $x \in [x_{i-1}, x_i]$.

(iii) The α_i and β_i arose as the constants of integration. To find them:

Take notes:

and so, for $i = 1, 2, \ldots, N$

$$\alpha_i = \frac{f_i}{h} - \frac{h}{6}\sigma_i, \quad \beta_i = \frac{f_{i-1}}{h} - \frac{h}{6}\sigma_{i-1}. \quad (2.3)$$

Notice that this gives $\beta_i = \alpha_{i-1}$.

(iv) So, now we "just" need the equations for σ_i. Two of these come from the fact that this is a "natural" cubic spline, with $S''(x_0) = 0$ and $S''(x_N) = 0$, therefore $\sigma_0 = 0$ and $\sigma_N = 0$.

The remaining $N - 1$ equations come from the fact that S' is continuous at $x_1, x_2, \ldots, x_{N-1}$. So we set that

$$s_i(x_i) = s_{i+1}(x_i).$$

After some work (see Exercise 2.3), we can show this means that the system is:

$$\sigma_0 = 0, \quad (2.4a)$$

$$\frac{1}{6}(\sigma_{i-1} + 4\sigma_i + \sigma_{i+1}) = \frac{1}{h^2}(f_{i-1} - 2f_i + f_{i+1}) \quad (2.4b)$$

for $i = 1, \ldots, N - 1$,

$$\sigma_N = 0. \quad (2.4c)$$

2.2.2 A cubic spline example

Example 2.2.2. Find the natural cubic spline interpolant to f at the points $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$ where $f_0 = 0, f_1 = 2, f_2 = 1$ and $f_3 = 0$.

[We won't do all the details in class, so here they are]

Solution:

From (2.2) we see we are looking for $S(x)=$

$$\begin{aligned}
&\alpha_1 x + \beta_1(1-x) + \frac{\sigma_0}{6h}(1-x)^3 + \frac{\sigma_1}{6h}x^3, x \in [0,1],\\
&\alpha_2(x-1) + \beta_2(2-x) + \frac{\sigma_1}{6h}(2-x)^3 + \frac{\sigma_2}{6h}(x-1)^3, x \in [1,2],\\
&\alpha_3(x-x_2) + \beta_3(3-x) + \frac{\sigma_2}{6h}(3-x)^3 + \frac{\sigma_3}{6h}(x-2)^3, x \in [2,3].
\end{aligned}$$

We first solve for the σ_i using (2.4):

$$\frac{1}{6}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -18 \\ 0 \\ 0 \end{pmatrix}$$

This gives

$$\sigma_0 = 0, \quad \sigma_1 = -\frac{24}{5}, \quad \sigma_2 = \frac{6}{5}, \quad \sigma_3 = 0.$$

Now use (2.3) to get

$$\alpha_1 = \frac{14}{5}, \quad \alpha_2 = \frac{4}{5}, \quad \alpha_3 = 0.$$

and

$$\beta_1 = 0 \quad \beta_2 = \frac{14}{5}, \quad \beta_3 = \frac{4}{5}.$$

The answer is

$$S(x) = \begin{cases} \frac{14}{5}x - \frac{4}{5}x^3, & x \in [0,1], \\ \frac{4}{5}(x-1) + \frac{14}{5}(2-x) - \\ \qquad \frac{4}{5}(2-x)^3 + \frac{1}{5}(x-1)^3, & x \in [1,2], \\ \frac{4}{5}(3-x) + \frac{1}{5}(3-x)^3. & x \in [2,3]. \end{cases}$$

This is shown in Figure 2.4.

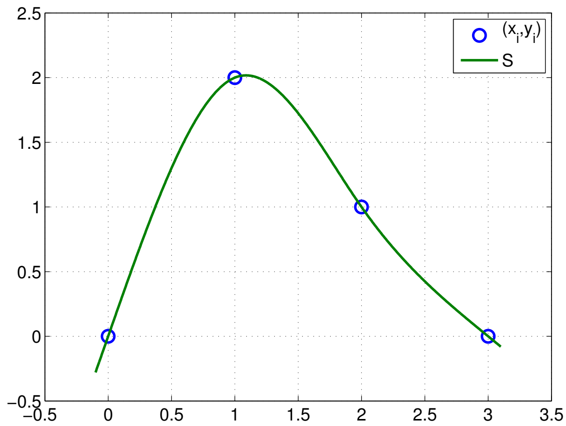

Fig. 2.4: Spline interpolant from Example 2.2.2

2.2.3 Error Estimates

We state the following error estimates without proof. You don't need to know these, or be able to prove them. But you should be able to use them if required.

Theorem 2.2.3. *If $f \in C^4[a,b]$ and S is its cubic spline interpolant on $N+1$ equally spaced points, then*

$$\|f - S\|_\infty \leqslant \frac{5}{384} M_4 h^4,$$
$$\|f' - S'\|_\infty \leqslant \frac{1}{24} M_4 h^3,$$
$$\|f'' - S''\|_\infty \leqslant \frac{3}{8} M_4 h^2,$$

where $M_4 = \|f^{(iv)}\|_\infty$.

Example 2.2.4. Give an upper bound on the error for the cubic spline interpolant to $f = e^x$ on the interval $[-1,1]$ with $N = 10$ mesh points.

Take notes:

Example 2.2.5. What is the smallest value of N that you must take to ensure that, if interpolating $f(x) = e^x$ on N equally sized intervals, the error is less that 10^{-8}? How does this compare with a linear spline interpolation (see Example 2.1.5)?

Take notes:

Recall Theorem 2.1.9 for linear splines. There is an analogous result for natural cubic splines.

Theorem 2.2.6. *Let u be any function that interpolates f at ω^N, and is such that $u \in H^2(x_0, x_N)$. Then*

$$\|S''\|_2 \leqslant \|u''\|_2.$$

The proof is not hard - it is analogous to the proof of Theorem 2.1.9.

2.2.4 Exercises

Exercise 2.3. When deducing the system of equations for the natural cubic spline, we showed how to construct the formulation given in (2.2) and the relationship between σ_i, α_i and β_i in (2.3). Now carefully show to deduce the system (2.4).

Exercise 2.4. (For students who did MA385). Write the equations in (2.4) as a linear system $A\boldsymbol{\sigma} = \mathbf{b}$, where A is an $n \times n$ matrix. Show that A is nonsingular, and hence that the system as a unique solution.

Exercise 2.5. Find the natural cubic spline interpolant to $f(x) = \sin(\pi x/2)$ at the nodes $\{x_i\}_{i=0}^3 = \{0,1,2,3\}$.

Calculate value of the interpolant at $x = 2.5$. What is the error at this point?

Exercise 2.6. Take $f(x) = \ln(x)$, $x_0 = 1$, $x_N = 2$. What value of N would you have to take to ensure that $|\ln(x) - S(x)| \leqslant 10^{-4}$ for all $x \in [1,2]$?

Exercise 2.7. (Loosely based on Burden and Faires, Q3.4.7) Suppose that S is a natural cubic spline on $[0,2]$ with

$$S(x) = \begin{cases} -3x + 2(1-x) + a(1-x)^3 + \frac{2}{3}x^3, & x \in [0,1), \\ b(2-x) + c(2-x)^3 + d(x-1)^3, & x \in [1,2]. \end{cases}$$

Find a, b, c, and d.