2.3 Piecewise Hermite Interpolation

In this section we introduce another type of cubic spline. It is not as smooth as the natural spline of the precious section—only it and its first derivative are continuous on $[x_0, x_N]$—and it requires that we know $f'(x_i)$. But its easier to construct (don't have to solve a linear system) and analyse than that natural spline.

As before, for short-hand, we write $f(x_i)$ as f_i and $f'(x_i)$ as f'_i.

And, as ever, we'll simplify the analysis by taking the points to be equally spaced: $x_i - x_{i-1} = h$ for each i).

2.3.1 PCHIP

Definition 2.3.1. Given a set of interpolation points $x_0 < x_1 < \cdots < x_N$, the *Piecewise Cubic Hermite Spline Interpolant* (*PCHIP*), S, to the function f, satisfies

(i) $S \in C^1[x_0, x_N]$,

(ii) $S(x_i) = f(x_i)$ and $S'(x_i) = f'(x_i)$ for $i = 0, 1, \dots, N$.

(iii) On each interval $[x_{i-1}, x_i]$, S is a cubic polynomial.

We can construct these splines as follows. On each interval $[x_{i-1}, x_i]$, let S be the cubic polynomial S_i given by

$$S_i(x) = c_0 + c_1(x - x_{i-1}) + c_2(x - x_{i-1})^2 + c_3(x - x_{i-1})^3, \quad (2.5)$$

Then we can show... <u>`Take notes`</u>:

This gives that

$$\begin{aligned} c_0 &= f_{i-1}, \\ c_1 &= f'_{i-1}, \\ c_2 &= \frac{3}{h^2}(f_i - f_{i-1}) - \frac{1}{h}(f'_i + 2f'_{i-1}), \\ c_3 &= \frac{1}{h^2}(f'_i + f'_{i-1}) - \frac{2}{h^3}(f_i - f_{i-1}). \end{aligned}$$

Figure 2.5 shows some PCHIP interpolants to $f(x) = 1/(1+x^2)$ on the interval $[-5, 5]$. Compare with Figures 2.1 and 2.3.

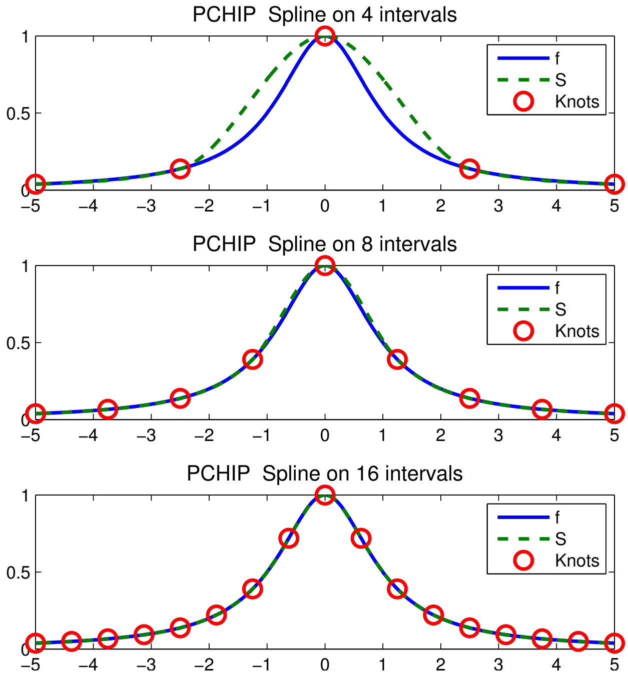

Fig. 2.5: PCHIP interpolants to $1/(1+x^2)$. Compare with Figure 2.3

We now want to prove an error estimate. The norm we use is

$$\|f - S\|_\infty := \max_{x_0 \leqslant x \leqslant x_N} |f(x) - S(x)|$$

Theorem 2.3.2. *Let $f \in C^4[x_0, x_N]$ and let S be the Hermite Cubic Spline interpolant to it at the N equally spaced points $a = x_0 < x_1 < \cdots < x_N = b$. Then*

$$\|f - S\|_\infty \leqslant \frac{h^4}{384}\|f^{(iv)}\|_\infty.$$

<u>`Take notes`</u>:

2.3.2 Estimating derivatives

You can ignore this section. It is included for completeness. It should make sense to anyone who took MA385, given that it relies only on Taylor series.

As we defined the PCHIP interpolant one needs to know f' in order to be able to construct it. However, you might notice that the MATLAB function `>> pchip` only requires f, not f'. How does it do this, one might wonder?

It turns out that there are important variants on the PHCIP scheme that don't involve knowing f'_0, f'_1, ..., f'_N, but instead uses *approximations* for f'. There are several approaches for deriving these estimates, including

(i) Geometrically. Recall, all we are really looking for is the slope of the tangent to f at $x = x_i$.

(ii) By finding a polynomial that interpolates f and differentiating that.

(iii) Taylor's Theorem.

For more details see [S96, Lecture 24]. Or look back at your notes for MA385, Lecture 14.

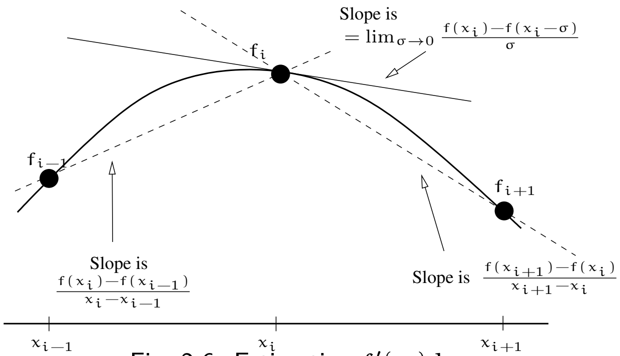

Fig. 2.6: Estimating $f'(x_i)dx$

As suggested by Figure 2.6 one could take

$$f'(x_i) \approx \frac{f_i - f_{i-1}}{x_i - x_{i-1}} = \frac{1}{h}(f_i - f_{i-1})$$

Or

$$f'(x_i) \approx \frac{f_{i+1} - f_i}{x_{i+1} - x_i} = \frac{1}{h}(f_{i+1} - f_i).$$

Since one seems to over estimate $f'(x_i)$, and the other seems to under-estimate, we could take the average:

$$f'(x_i) \approx \frac{-f_{i-1} + f_{i+1}}{x_{i+1} - x_{i-1}} = \frac{1}{2h}(-f_{i-1} + f_i)$$

These approximations can also be easily derived from Taylor's Theorem:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\eta),$$

for some $\eta \in (a, b)$. Using this we can get

$$f_{i+1} = f_i + hf'(x_i) + \frac{h^2}{2}f''(x_i) + \frac{h^3}{6}f'''(\tau_1),$$
$$f_{i-1} = f_i - hf'(x_i) + \frac{h^2}{2}f''(x_i) - \frac{h^3}{6}f'''(\tau_2),$$

for $\tau_1 \in (x_i, x_{i+1})$ and $\tau_2 \in (x_{i-1}, x_i)$. Now subtract the 1st equation from the second to get the formula and error estimate.

2.3.3 Exercises

Exercise 2.7. Recall Exercise 2.5. Calculate the value to the PCHIP interpolant to $f(x) = \sin(\pi x/2)$ at the nodes $\{x_i\}_{i=0}^3 = \{0, 1, 2, 3\}$ at the point $x = 2.5$. What is the error at this point?

Exercise 2.8. Let $f(x) = \ln(x)$. Let l and S be the piecewise linear and Hermite cubic spline interpolants (respectively) to f on $n+1$ equally spaced points $1 = x_0 < x_1 < \cdots < x_N = 2$. What value of n would you have to take to ensure that

(i) $\max_{1 \leqslant x \leqslant 2} |f(x) - l(x)| \leqslant 10^{-4}$?

(ii) $\max_{1 \leqslant x \leqslant 2} |f(x) - S(x)| \leqslant 10^{-4}$?

Also, using the code you developed in Lab 2, compare these theoretical results with the value needed on practice.

Exercise 2.9. There are ways of constructing the PCHIP, other than (2.5). For example, let $s = x - x_{k-1}$, then

$$S(x) = \frac{h^3 - 3hs^2 + 2s^3}{h^3}f_{k-1} + \frac{3hs^2 - 2s^3}{h^3}f_k + \frac{s(s-h)^2}{h^2}f'_{k-1} + \frac{s^2(s-h)}{h^2}f'_k$$

Show that this is the same as the PCHIP.