Chapter 3

Numerical Integration

3.1 Introduction

Problem: *Given a real-valued function* f *that is continuous on* $[a, b]$*, can we find an estimate for*

$$I(f) := \int_a^b f(x)dx?$$

And if we can, can we say how accurate it is?

Why bother?

- many problems in applicable mathematics require definite integrals to be evaluated.[1]
- evaluating them by finding the anti-derivative can be hard, and very hard to automate.
- some times, although the function is integrable, its anti-derivative doesn't exist in a closed form.

The process of numerically estimating a definite integral is called *Numerical Integration* or *Quadrature*.

The formulae we'll derive all look like

$$Q_n(f) := q_0 f(x_0) + q_1 f(x_1) + q_2 f(x_2) + \dots q_n f(x_n).$$

Here the points x_i are called *quadrature points* and the q_i are *quadrature weights*. So we need a way of choosing these. The simplest approach is to take the points to be equally spaced, i.e., $x_i = a + hi$ where $h = (b - a)/n$.

How to choose the weights? We've spent quite a while talking and thinking about approximating functions with polynomials. So why not find a polynomial interpolant to f and take the integral of that to be the answer? The appeal of this approach is due to the fact that

- Finding polynomial interpolants is easy.
- Integrating polynomials is easy.
- We can estimate the error easily (yet again, we'll make use of Theorem 1.3.3)

[1] These methods were originally motivated by problems in astronomy. They're credited to the English mathematician/astronomer Roger Cotes (1682–1716), working with Isaac Newton.

This leads to the Newton-Cotes methods, which are the subject of this section, and Section 3.2 Later again, we'll look at more sophisticated methods, called *Gaussian Methods* which use non-uniformly spaced points.

3.1.1 Newton-Cotes methods

Definition 3.1.1. The *Newton-Cotes* quadrature rule for $\int_a^b f(x)dx$ with $n + 1$ points is derived by integrating exactly the polynomial of degree n that interpolates f at the n equally spaced points $a = x_0 < x_1 < \dots < x_n = b$. The method is written as

$$Q_n(f) := q_0 f_0 + q_1 f_1 + q_2 f_2 + \dots q_n f_n.$$

That is, the quadrature weights are chosen so that

$$Q_n(f) = \int_a^b p_n(x)dx,$$

where p_n is the polynomial of degree n that interpolates f at the $n + 1$ quadrature points. However, it turns out that we can compute the weights $q_0, \dots, q_n$, *without* knowing p_n. We'll do this for $n = 1$ in the next section, and $n = 2$ (the most interesting case) in Section 3.2.

3.1.2 The Trapezium rule

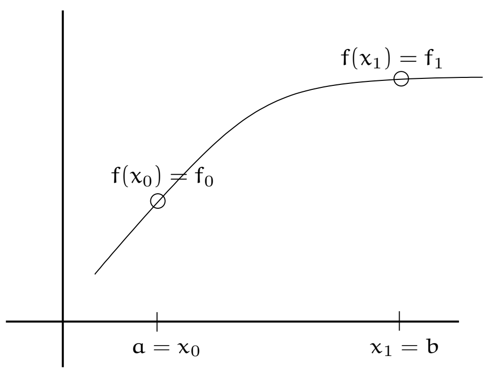

Fig. 3.1: Want to estimate $\int_a^b f(x)dx$

We want to estimate the integral of a function, f, on $[a, b]$, shown in Figure 3.1. Here are three approaches we could take.

Method 1: We could try to estimate the area of the trapezium the fits under the graph, as show in Figure 3.2. Clearly this comes to

$$Q_1(f) = (b-a)f_0 + (b-a)\frac{f_1 - f_2}{2} = (b-a)\frac{f_1 + f_0}{2}. \quad (3.1)$$

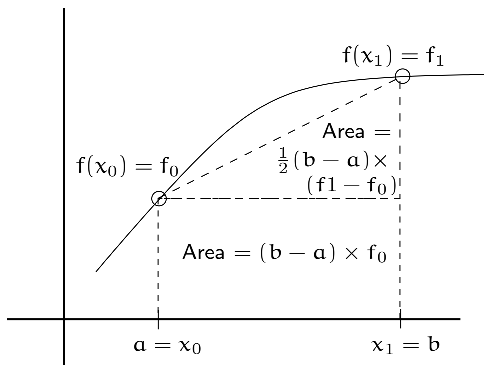

Fig. 3.2: Whats the area under the graph?

Method 2: As shown in Figure 3.3 we could find p_1, the polynomial of degree 1 that interpolates f at $x = a$ and $x = b$.

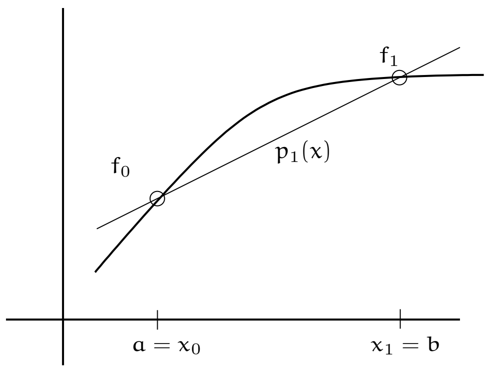

Fig. 3.3: Integrate $p_1(x)$ from x_0 to x_1

Take notes:

Note that this shows that $q_i = \int_a^b L_i(x)dx$, where, as usual, the L_i are the Lagrange Polynomials. (See §1.2.4).

Method 3: The third approach for generating the Trapezium Rule is called the *Method of Undetermined Coefficients*. Because the method is based on integrating a linear function we expect it to yield an exact solution for any constant or linear function (i.e., there should be no error). To keep the algebra simple, we'll take $a = 0$ and $b = 1$. So,

$$Q_1(f) = q_0 f(0) + q_1 f(1),$$

and, setting $f(x) \equiv 1$, and then $f(x) = x$ we get

Take notes:

Now we need to extend this to estimating $\int_a^b g(x)dx$ as follows:

Take notes:

Example 3.1.2. Use the Trapezium Rule to estimate

$$\int_0^{\pi/4} \cos(x)dx.$$

Calculate the (exact) error $|\int_a^b f(x)dx - Q_1(f)|$.

Take notes:

3.1.3 Exercises

Exercise 3.1. Let $q_0, q_1, \dots, q_n$ be the quadrature weights for the Newton-Cotes rule $Q_n(f)$. Show that $q_i = q_{n-i}$ for $i = 0, \dots n$.

Exercise 3.2. Show that $\sum_{i=0}^{n} q_i = b - a$.