3.2 Simpson's Rule

Next we'll consider the *3-point Newton-Cotes scheme* which is based on integrating the quadratic interpolant to $f(x)$.

Simpson's Rule

$$Q_2(f) = \frac{b-a}{6}\big(f(a) + 4f(\frac{a+b}{2}) + f(b)\big). \tag{3.2}$$

This is called *Simpson's Rule*[2]. To show how to derive it, we'll use the Method of Undetermined Coefficients again. First restrict our attention to approximating $\int_0^1 g(x)dx$. This method should be exact for all constant, linear and quadratic polynomials. Taking $g(x) \equiv 1$, $g(x) = x$ and $g(x) = x^2$ we get the set of equations

Take notes:

This is easily solved giving

$$\int_0^1 g(x)dx \approx \frac{1}{6}g(0) + \frac{2}{3}g(1/2) + \frac{1}{6}g(1). \tag{3.3}$$

To extend this to the interval $[a, b]$, we again use a change of variables to get the general Simpson's Rule (3.2) .

Example 3.2.1. Use Simpson's rule to estimate $\int_0^{\pi/4} \cos(x)dx$, and calculate the (exact) error $|\int_a^b f(x)dx - Q_2(f)|$.

Take notes:

[2]Thomas Simpson, 1710–1761. One of the most distinguished of a group of itinerant lecturers who taught in the London coffee-houses, Hutton (famous text-book writer) said of him

> *It has been said that Mr Simpson frequented low company, with whom he used to guzzle porter and gin: but it must be observed that the misconduct of his family put it out of his power to keep the company of gentlemen, as well as to procure better liquor.*

The method was known well before Simpson's time: it had been used by Cavalieri (a student of Galileo) in 1639, James Gregory, Johannes Kepler, and others.

3.2.1 Newton-Cotes error estimates

We'll now derive error estimates for general Newton-Cotes methods, and look at the specific cases of the Trapezium and Simpson's rules.

Theorem 3.2.2. *Let*

$$M_{n+1} := \max_{a \leqslant x \leqslant b} |f^{(n+1)}(x)|,$$

and $\pi_{n+1}(x)$ be the usual nodal polynomial. Define

$$\mathcal{E}_n := \Big|\int_a^b f(x)dx - Q_n(f)\Big|.$$

Then

$$\mathcal{E}_n \leqslant \frac{M_{n+1}}{(n+1)!}\int_a^b |\pi_{n+1}(x)|dx,$$

The proof just comes directly Cauchy's Theorem (Theorem 1.3.3).

Take notes:

Error estimates for the Trapezium Rule

Theorem 3.2.3. *For the Trapezium Rule (3.1)*

$$\mathcal{E}_1 \leqslant \frac{(b-a)^3}{12}M_2. \tag{3.4}$$

The proof is an exercise.

Example 3.2.4. Use (3.4) to get an upper bound on the error for the estimate of $\int_0^{\pi/4} \cos(x)dx$ using the Trapezium rule. How does this compare with the actual error that we found in Example 3.1.2?

Take notes:

Example 3.2.5. If use the Trapezium Rule to estimate the integral of x^2 on the interval $[0, 1]$ we get

$$\int_0^1 x^2 dx = \frac{1}{3} \qquad \text{and} \qquad Q_1(x^2) = \frac{1}{2}(0+1) = \frac{1}{2}.$$

So the error is $1/6$, exactly as the theory predicts.

Error estimates for Simpson's rule

One could also use Theorem 3.2.2 to show that, for Simpson's Rule,

$$\mathcal{E}_2 \leqslant \frac{(b-a)^4}{196}M_3, \tag{3.5}$$

but don't bother because, although correct, it is not *sharp* (that is, it is pessimistic).

Example 3.2.6. Use (3.5) to get an upper bound on the error for the estimate of $\int_0^{\pi/4} \cos(x)dx$ using **Simpson's** rule. How does this compare with the actual error that we found in Theorem 3.2.1?

`Take notes`:

Example 3.2.7. We expect Simpson's Rule to give *exactly* the right answer for integrals of constant, linear and quadratic functions. If we take $f(x) = x^3$, $a = 0$ and $b = 1$, then formula above suggests that (approx) $\mathcal{E}_2 \leqslant 0.03$. But

`Take notes`:

3.2.2 Exercises

Exercise 3.3. Deduce the 4-point Newton-Cotes Rule for estimating the integral $\int_0^1 f(x)dx$:

$$Q_3(f) = q_0 f(x_0) + q_1 f(x_1) + q_2 f(x_2) + q_3 f(x_3).$$

Extend the rule to estimate the integral of functions over $[a, b]$.

Exercise 3.4. Prove the error bound given for the Trapezium rule in (3.4).