3.3 Precision and Composition

3.3.1 Precision

We know from Example 3.2.7 that Simpson's Rule applied to approximating $\int_0^1 x^3 dx$ yields exactly the right answer (i.e., the error is zero).

We now claim that Simpson's Rule is exact for *any* polynomial of degree 3 or less.

In class to simplified by taking $a = -1$ and $b = 1$. Here are the **details** for the general case. Denote by $Q_2(f)$ the approximation of $\int_a^b f(x)dx$ with Simpson's Rule:

$$Q_2(f) = \frac{b-a}{6}\Big(f(a) + 4f(\frac{a+b}{2}) + f(b)\Big).$$

Since the method can be derived by integrating the quadratic that interpolates $f(x)$ at the three points a, $(a+b)/2$, and b, it is clearly exact for all quadratics. (We also know this from Theorem 3.2.3).

Let $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3$. Then

$$\int_a^b f(x)dx = \int_a^b (c_0 + c_1x + c_2x^2)\,dx + c_3\int_a^b x^3 dx = \int_a^b (c_0 + c_1x + c_2x^2)\,dx + c_3\frac{b^4-a^4}{4}.$$

Also,

$$Q_2(f) = Q_2(c_0 + c_1x + c_2x^2) + Q_2(c_3x^3) = \int_a^b (c_0 + c_1x + c_2x^2)\,dx + c_3(\frac{b-a}{6})\Big(a^3 + 4(\frac{a+b}{2})^3 + b^3\Big). \quad (3.6)$$

With a bit of symbolic manipulation we get that

$$\frac{b^4-a^4}{4} = (\frac{b-a}{6})\Big(a^3 + 4(\frac{a+b}{2})^3 + b^3\Big),$$

as required.

In (3.5) we gave the result of a nïeve attempt to derive an upper bound for the error in Simpson's rule:

$$\mathcal{E}_2 \leqslant \frac{(b-a)^4}{196}M_3.$$

This is not wrong – just not sharp. For example it does not give that Simpson's Rule is exact for all cubics. The sharp result is

Theorem 3.3.1.

$$|E_2(x)| = |\int_a^b f(x)dx - Q_2\big(f(x)\big)| \leqslant \frac{(b-a)^5}{2880}M_4.$$

For the proof see the text book (Theorem 7.2 of [SM03]). Instead of working through it in class we'll prove a more general version of a consequence of Theorem 3.3.1.

Definition 3.3.2 (Precision of a Quadrature Rule)**.** A quadrature rule has *precision* n if it is exact for all polynomials of degree n or less. That is, the rule $Q(f)$ has precision n if

$$Q(p_n) = \int_a^b p_n(x)dx \quad \text{for all } p_n \in \mathcal{P}_n.$$

Example 3.3.3. By construction, the $(n+1)$-point Newton-Cotes rule has precision n.

Theorem 3.3.4. *If $Q_{2k}(\cdot)$ is a Newton-Cotes quadrature rule on $2k+1$ points, then $Q_{2k}(\cdot)$ has in fact precision $2k+1$.*

Proof: Let p_{n+1} be a polynomial of degree $n+1$. We wish to show that $Q_n(p_{n+1}) = \int_a^b p_{n+1}(x)dx$. We can take $a = -1$, $b = 1$ because a simple linear transformation can be used to map to an arbitrary interval.

Also, since the quadrature points are equally spaced on $[-1, 1]$ we have that $x_i = -x_{n-i}$.

Furthermore (see Exercise 3.1), the quadrature weights are symmetric: $q_i = q_{n-i}$.

Take notes:

3.3.2 Composite Rules

Suppose that we want to estimate $\int_a^b f(x)dx$ and the Trapezium rule is not sufficiently accurate. We could try Simpson's Rule, which should be better. Failing that, we could try a Newton-Cotes rule that uses more points. However, quite apart from the fact that it might be tedious to derive these rules, such schemes are based on poynomial interpolation, and we know (Runge's example again!) that high-order polynomial interpolation can be very inaccurate. See Exercise 3.5

It is better to use a *Composite Rule*. This is analogous to the piecewise polynomial interpolation introduced in Section 2.1. For the Composite Trapezium Rule, as shown in Figure 3.4, we divide $[a, b]$ into N intervals of size $h = (b - a)/N$. Applying the Trapezium Rule on each interval $[x_{i-1}, x_i]$ we get

$$\int_{x_{i-1}}^{x_i} f(x)dx \approx h\frac{f_{i-1} + f_i}{2}.$$

Summing for the n intervals we get

$$\int_a^b f(x)dx \approx \frac{b-a}{N}\Big(\frac{f_0}{2} + f_1 + f_2 + \cdots + f_{N-1} + \frac{f_N}{2}\Big). \quad (3.7)$$

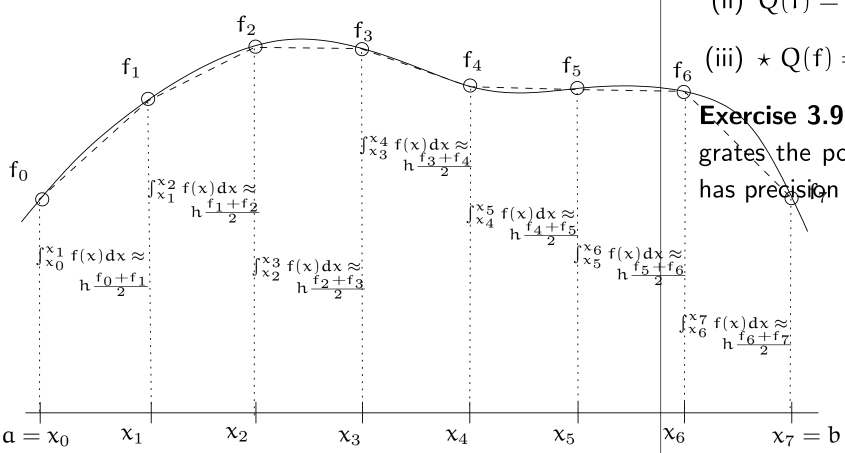

Fig. 3.4: Estimating $\int_a^b f(x)dx$ using the Composite Trapezium Method

3.3.3 Exercises

Exercise 3.5. Explain clearly, with an example, why in general it is not true that $Q_n(f) \to \int_a^b f(x)dx$ as $n \to \infty$.

Exercise 3.6.

(i) Use Theorem 3.2.2 to deduce an error estimate for the Composite Trapezium Rule (3.7).

(ii) Taking $N = 10$, give an upper bound for the error in the Composite Trapezium Rule when approximating $\int_1^2 \ln(x)dx$.

(iii) What value of n would you have to take to ensure that the error was less that 10^{-5}?

Exercise 3.7.

(i) Deduce the formula for the *composite Simpson's Rule*, and use Theorem 3.3.1 to derive an error estimate.

(ii) What value of N would you have to take to ensure that the error in the estimate of $\int_1^2 \ln(x)dx$ is less that 10^{-6}?

(iii) Denote the $(N+1)$-point Composite Simpson's Rule by $S_N(f) \approx \int_a^b f(x)dx$. Show that, for sufficiently smooth $f(x)$,

$$\lim_{n\to\infty} S_N(f) = \int_a^b f(x)dx.$$

Exercise 3.8. Determine the precision of the following schemes for estimating $\int_0^1 f(x)dx$.

(i) $Q(f) = f(\frac{1}{2})$.

(ii) $Q(f) = \frac{1}{4}f(0) + \frac{3}{4}f(\frac{2}{3})$.

(iii) $\star$ $Q(f) = \frac{3}{2}f(\frac{1}{3}) - 2f(\frac{1}{2}) + \frac{3}{2}f(\frac{2}{3})$.

Exercise 3.9. Suppose that a quature rule exactly integrates the polynomials 1, x, x^2, $\dots$, x^n. Show that it has precision n.