3.4 Gaussian Quadrature

3.4.1 Introduction

To date we have found some numerical schemes that approximate $\int_a^b f(x)dx$ as the weighted average of values of f at $n+1$ equally spaced points. These methods have precision n (or $n+1$ in special cases).

With Gaussian Quadrature [3] we choose both the quadrature weights and points in such as way as to maximize the precision of the method. There are three equivalent approaches to doing this.

(i) *Undetermined Coefficients*: The obvious way for, say, $n = 2$. But, unlike Newton-Cotes, we have to solve a system of *nonlinear* equations. Even for $n = 3$ this can become difficult.

(ii) *Base the method on integrating the Hermite Interpolant of the integrand, f (see §1.5)*, and choose the points so that the coefficients of $f'(x_i)$ are zero. This approach is the easiest to analyse, but less useful for construction.

(iii) *Finding the zeros of the members of a sequence of orthogonal monic polynomials*. This is the approach we will emphasise most, as it gives us an easy way of proving the precision of the methods.

This section is perhaps the most mathematically rich in the course. I'd encourage you to read further: Chapter 10 of Süli and Mayers [SM03] is devoted to this. However, much of the basic theory is developed in Section 9.4 on Orthogonal Polynomials. See also [S96, Lectures 22 and 23] and [SB92, §3.6].

3.4.2 Undetermined Coefficients

Example 3.4.1. *Find a two point rule*

$$\int_{-1}^{1} f(x)dx \approx G_1(f) := w_0 f(x_0) + w_1 f(x_1),$$

that is exact for all polynomials of degree 3 or less.

This leads a the non-linear system which we must solve

<u>Take notes</u>:

That is

$$\int_{-1}^{1} f(x)dx \approx G_1(f) := f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right). \qquad (3.8)$$

Example 3.4.2. Let $f(x) = \exp(-x)$. If we estimate $\int_{-1}^{1} f(x)dx$ using each of the Trapezium Rule, Simpson's Rule and the Gaussian Rule above we find the errors are, respectively, 0.735, 0.01165 and 0.00771. Using the composite trapezium rule, we find we would have to take $N = 11$ to obtain an estimate that is more accurate than the two-point Gaussian Rule.

Example 3.4.3. If you use the $G_1(\cdot)$ rule to estimate $\int_0^{\pi/4} \cos(x)dx$ you'll find that

$$\left|\int_0^{\pi/4} \cos(x)dx - G_1(\cos)\right| = \left|\frac{1}{\sqrt{2}} - 0.07070432596\right| \approx 6.35 \times 10^{-5}.$$

Compare with results for the same problem when

- The trapezium rule is used (Theorem 3.1.2).
- Simpson's rule is used (Theorem 3.2.1).

Example 3.4.4. A variation of Gaussian quadrature is *constrained Gaussian quadrature*, also known is the *Gauss-Lobatto method*. Rather than allowing all of the quadrature points to vary in order to maximize the precision of the method, we fix some of them — usually the end points. Use undetermined coefficients to derive the 3-point *Gauss-Lobatto rule*:

$$\int_{-1}^{1} f(x)dx \approx w_0 f(-1) + w_1 f(x_1) + w_2 f(1),$$

where we fixed $x_0 = 0$ and $x_2 = 1$.

<u>Take notes</u>:

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[3] 

Johann Carl Friedrich Gauß, born 1777 in Braunschweig, died 1855 in Göttingen.

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3.4.3 Exercises

Exercise 3.10. Use a change of variables, as we did with the Trapezium rule, to show that the rule for approximating $\int_0^1 f(x)dx$ is

$$G_1(f) = \frac{1}{2}\left(f(\frac{1}{2} - \frac{1}{2\sqrt{3}}) + f(\frac{1}{+}\frac{1}{2\sqrt{3}})\right).$$

More generally, extend the $G_1(f)$ rule in (3.8) to an arbitrary interval $[a, b]$.

Exercise 3.11. Use $G_1(x)$ to estimate $\int_1^2 \ln(x)dx$. How does this compare with the Trapezium and Simpson's Rule?

Exercise 3.12. Derive a 3-point Gaussian Quadrature Rule to estimate $\int_{-1}^1 f(x)dx$. *Hint:* $x_1 = 0$.