3.5 Orthogonal Polynomials

High order Newton-Cotes methods are of little use because of the problems associated with interpolation be high degree polynomials at equally spaced points. However, high-order Gaussian methods are very useful.

Driving such methods by undetermined coefficients is not practical, however. There is a simpler way, but some mathematical preliminaries are required.

3.5.1 Inner products

Definition 3.5.1 (Vector Space)**.** V is a *vector space* (a.k.a., a *linear space*) over a field F (e.g, the real or complex numbers) if for all $u, v, w \in V$ and $a, b \in F$:

(i) $u + v \in V$ (closed under addition)

(ii) $u + v = v + u$ (Commutativity)

(iii) $(u + v) + w = u + (v + w)$ (Associativity)

(iv) V has a zero vector $\mathbf{0}$ such that $u + \mathbf{0} = u$.

(v) $-u \in V$

(vi) $au \in V$

(vii) $a(bu) = (ab)u$

(viii) F contains 0 and 1 such that $1u = u$, $0u = \mathbf{0}$.

(ix) $a(u + v) = au + av$, and $(a + b)u = au + bu$.

Examples:

Definition 3.5.2 (Inner Product)**.** Let V is a real vector space. An **Inner Product** (IP) is a real-valued function $(\cdot, \cdot)$ on $V \times V$ such that, for all $f, g, h \in V$,

(i) $(f + g, h) = (f, h) + (g, h)$,

(ii) $(\lambda f, g) = \lambda(f, g)$, for $\lambda \in \mathbb{R}$.

(iii) $(f, g) = (g, f)$,

(iv) $(f, f) \geqslant 0$. $\quad (f, f) = 0 \iff f \equiv 0$.

Example 3.5.3. Let $\mathbb{R}^n$ be our vector space, with $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$. Then

$$(x, y) := \sum_{i=1}^{n} x_i y_i,$$

is an inner product.

Example 3.5.4. The set of real-valued functions that are continuous and defined on the ionterval $[a, b]$, denoted $C[a, b]$, is a vector space. And

$$(f, g) := \int_a^b f(x)g(x)dx, \tag{3.9}$$

is an inner product on this space.

We could consider the more general family "weighted" inner products:

$$(f, g) := \int_a^b w(x)f(x)g(x)dx,$$

where $w(x)$ is some positive "weight function." However the IP we will use for the remainder of this chapter is always the one defined in (3.9). That is, we are just concerned with the case $w(x) \equiv 1$.

3.5.2 A Sequence of Orthogonal Monic Polynomials

(Please see [S96, Lecture 23] for more details).

Definition 3.5.5 (Monic Polynomial)**.** A polynomial is *monic* if the coefficient of its leading term is 1.

Example 3.5.6.

Take notes:

Definition 3.5.7. Two elements a, b, of a vector space are *orthogonal* with respect to a given inner product $(\cdot, \cdot)$ if $(a, b) = 0$.

Example 3.5.8. Take notes:

Example 3.5.9. Take the space of polynomials of degree 2 or less and the IP

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx.$$

Let $p(x) \equiv 1$, $q(x) \equiv x$, $r(x) \equiv x^2 - 1/3$, and $f(x) = 3x - 4$, then:
Take notes:

3.5.3 A sequence of orthogonal polynomials

As given in Definition 3.5.5, a polynomial is *monic* if the coefficient of the leading term is 1:

$$p_n = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-1} + \cdots + c_1 x + c_0.$$

We'll now look at a sequence of such polynomials $\{\widetilde{p}_0, \widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n, \ldots\}$ that have the property they are orthogonal to each other:

$$(\widetilde{p}_i, \widetilde{p}_j) := \int_a^b \widetilde{p}_i(x)\widetilde{p}_j(x)dx = 0 \quad \text{if } i \neq j.$$

We want to establish some important facts about monic polynomials:

- A set of monic polynomials of degrees 1, 2, ..., n, forms a basis for $\mathcal{P}_n$.
- If the members of that set are orthogonal to each other, then they are orthogonal to *all* polynomials of lower degree.
- We can construct such as set.

Lemma 3.5.10. *Let $\{p_i\}_{i=0}^n$ be a sequence of polynomials where each p_i is monic an* exactly *of degree i. This sequence forms as basis for $\mathcal{P}_n$.*

Proof:
Take notes:

This lemma means that if q is a polynomial of degree n then it can be written uniquely as a linear combination of the p_i:

$$q(x) = \sum_{i=0}^{n} a_i p_i(x),$$

for some unique choice of the real coefficients a_i.

Definition 3.5.11. The sequence $\{\widetilde{p}_i\}_{i=0}^n$ is a sequence of *monic*, *orthogonal* polynomials if each $\widetilde{p}_i$ is monic and *exactly* of degree i and

$$(\widetilde{p}_i, \widetilde{p}_j) = 0 \quad \text{if } i \neq j.$$

Lemma 3.5.12. *If $\widetilde{p}_j \in \{\tilde{p}_i\}_{i=0}^{\infty}$ then $\widetilde{p}_j$ is orthogonal to all polynomials of degree less than j.*

Proof:
Take notes:

3.5.4 Constructing the Sequence

Theorem 3.5.13. *The sequence $\{\tilde{p}_i\}_{i=0}^{\infty}$ exists and can be constructed as follows: Let α and β be defined as*

$$\alpha_{n+1} = \frac{(x\widetilde{p}_n, \widetilde{p}_n)}{(\widetilde{p}_n, \widetilde{p}_n)}, \quad \textit{and} \quad \beta_{n+1} = \frac{(x\widetilde{p}_n, \widetilde{p}_{n-1})}{(\widetilde{p}_{n-1}, \widetilde{p}_{n-1})},$$

then the sequence is given by

$$\widetilde{p}_0(x) \equiv 1, \qquad \widetilde{p}_1(x) = x - \alpha_1$$

and

$$\widetilde{p}_{n+1}(x) = (x - \alpha_{n+1})\widetilde{p}_n(x) - \beta_{n+1}\widetilde{p}_{n-1}(x),$$

for $n \geqslant 1$.

The proof uses *Gram-Schmidt Orthogonalization*.

Take notes:

Example 3.5.14. If we use the inner product $(f,g) := \int_0^1 f(x)g(x)$ then the first 4 polynomials in the sequence are:

Take notes:

Example 3.5.15. The zeros of $\widetilde{p}_2$ are ...

Take notes:

3.5.5 Properties of the sequence

One of the ways of constructing Gaussian Quadrature rule $G_n(\cdot)$ on $n+1$ is to take the quadrature points as the roots of $\widetilde{p}_{n+1}$. We know (from the fundamental theorem of algebra) a polynomial of degree $n+1$ has exactly $n+1$ roots in $\mathbb{C}$ up to multiplicity.

However, the polynomials $\widetilde{p}$ have the special properties, established by the following result.

Lemma 3.5.16. *Let* $\widetilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty} = \{\widetilde{p}_0, \widetilde{p}_1, \ldots,\}$ *be the set of monic polynomials that are orthogonal with respect to the (usual) inner product.*

(i) The zeros of each $\widetilde{p}_i \in \{\tilde{p}_i\}_{i=0}^{\infty}$ *are simple (not repeated).*

(ii) All the zeros of $\widetilde{p}_i$ *are real numbers in the interval* $[a,b]$.

(*A slightly different proof of these facts are given in [SM03, Thm 9.4].*

Take notes:

3.5.6 Exercises

Exercise 3.13. Suppose that $(\cdot,\cdot)$ is an inner product. Show that $\|u\| := \sqrt{(u,u)}$ is a norm.

Exercise 3.14. $\mathcal{P}_n$, the space of polynomials of degree (at most) n forms a vector space. Is it true that the space of *monic* polynomials of degree n forms a vector space?

Exercise 3.15. (i) Using the Inner Product

$$(f,g) := \int_{-1}^{1} f(x)g(x)dx,$$

find $\widetilde{p}_0(x)$, $\widetilde{p}_1(x)$, $\widetilde{p}_2(x)$ and $\widetilde{p}_3(x)$.

(ii) Find the zeros of $\widetilde{p}_2(x)$ and call them x_0 and x_1. Construct a quadrature rule for $\int_{-1}^{1} f(x)dx$ taking these as the quadrature points, and the weights as the integrals to the corresponding Lagrange polynomials. Verify that this is the same rule as given in (3.8).

(iii) Repeat this exercise using the zeros of $\widetilde{p}_3(x)$ as the quadrature points. Verify that the rule you get is the same as Exercise 3.12.