

## 3.6 Gaussian Quadrature via Orthogonal Polynomials

At the start of this section, we introduced the using Gaussian Quadrature technique for estimating integrals

$$\int_a^b f(x) dx \approx G_n(f) := \sum_{k=0}^n w_k f(x_k),$$

where the points  $x_k$  and weights  $w_k$  are chosen to maximise the precision of the method.

We now have an equivalent way of defining the method,  $G_n(\cdot)$ , and deriving the coefficients:

- (a) Write down the set of monic polynomials  $\{\tilde{p}_0, \tilde{p}_1, \dots, \tilde{p}_{n+1}\}$  that is orthogonal with respect to the inner product

$$(u, v) := \int_a^b u(x)v(x) dx.$$

Note that  $a$  and  $b$ , the limits of integration for the IP, are the same as for the integral we are trying to estimate.

- (b) From Theorem 3.5.16, we know that  $\tilde{p}_{n+1}$  has  $n+1$  zeros in the interval  $[a, b]$ . Let these be the quadrature points of the method.<sup>4</sup>

For example, the polynomial  $\tilde{p}_5 = x^5 - (10/9)x^3 + (5/21)x$  is shown in Figure 3.5, with its zeros highlighted.

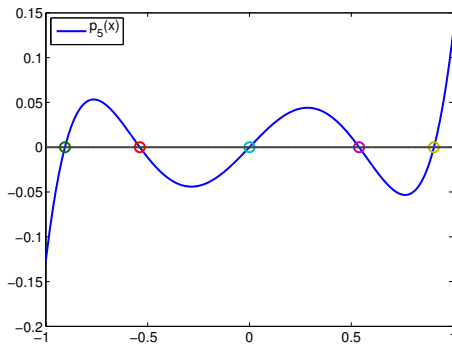


Fig. 3.5: The polynomial  $\tilde{p}_5 = x^5 - (10/9)x^3 + (5/21)x$

- (c) Take the quadrature weights to be  $w_k = \int_a^b L_k(x) dx$ , where the  $L_k$  are the usual Lagrange polynomials for this set of points.

The key property of this method is stated in the following theorem.<sup>5</sup>

<sup>4</sup>In practice one would take a simpler IP such as

$$(u, v) = \int_{-1}^1 u(x)v(x) dx,$$

and then transform the zeros from  $[-1, 1]$  to  $[a, b]$  using a linear transformation.

<sup>5</sup>In an earlier version of these notes, this was number Theorem 3.6.16

**Theorem 3.6.1.** Let  $x_0, \dots, x_n$  to be the zeros of  $\tilde{p}_{n+1}$ , the  $(n+1)$ th polynomial in the sequence of orthogonal monic polynomials  $\{\tilde{p}_i\}_{i=0}^\infty$ . Set

$$G_n(f) = w_0 f(x_0) + \dots + w_n f(x_n) \quad \text{where } w_i = \int_a^b L_i(x) dx. \quad (3.10)$$

Then  $G_n(f)$  has precision  $2n+1$ .

The proof of this is an exercise.

### 3.6.1 Error estimates

Our final task associated with numerical integration is to prove that, as  $n \rightarrow \infty$ , so  $G_n(f) \rightarrow \int_a^b f(x) dx$ .

We are not going to do this in full detail, but the key ideas will be presented.

We would like to prove the following error estimate, which is closely related to Theorem 1.5.2 (error in Hermite interpolation):

**Theorem 3.6.2.**

$$\int_a^b f(x) dx - G_n(f) = \frac{f^{(2n+2)}(\tau)}{(2n+2)!} \int_a^b \pi_{n+1}(x)^2 dx.$$

The trick here is that we are claiming that Gaussian Quadrature is the same as integrating the Hermite interpolant to  $f$ , even though the latter depends on  $f'(x_k)$  although the former does not.

Moreover, we would like to show that

$$G_n(f) \rightarrow \int_a^b f(x) dx \text{ as } n \rightarrow \infty.$$

To do this we need to establish several facts. In each case we make use of Theorem 3.6.1, a consequence of which is that, if  $f$  is a polynomial of degree at most  $2n-1$ , then

$$\int_a^b f(x) dx = \sum_{k=0}^n w_k f(x_k).$$

First, recall the basis functions that we use for Hermite Interpolation Section 1.5:

$$H_i(x) = [L_i(x)]^2 (1 - 2L'_i(x_i)(x - x_i)),$$

$$K_i(x) = [L_i(x)]^2 (x - x_i).$$

Take notes:

Now we can deduce Theorem 3.6.2.

Take notes:

Next we want to show that each of the  $w_k$  are positive. From (3.10) we have that the Gaussian Quadrature weights are  $w_k = \int_a^b L_k(x) dx$  where the  $L_k$  are the usual Lagrange Polynomials:

$$L_k(x_j) = \begin{cases} 1 & k = j \\ 0 & k \neq j, \end{cases}$$

associated with the Gaussian interpolation points. Then in fact

Take notes:

and so that the  $w_k$  are all positive. It follows directly that  $0 < w_k < (b - a)$ , for  $k = 0, 1, 2, \dots, n$ :

Take notes:

### 3.6.2 Convergence

Section 10.4 of [SM03] also covers this, though from a different angle. One of the most interesting aspects of this theory is given in Theorem 10.2:

**Theorem 3.6.3.**

$$\lim_{n \rightarrow \infty} G_n(f) = \int_a^b f(x) dx.$$

(Here we'll just given an out line of the proof. You are not required to know this for the exam).

The Weierstrass approximation theorem, which we mentioned back in Section 1.6.1, tells us that, for any  $\epsilon > 0$ , there exists a polynomial  $p$  such that  $|f(x) - p(x)| \leq \epsilon$ . Let  $n$  be the degree of this polynomial. Let  $G_n(\cdot)$  be the  $n + 1$  point Gaussian Quadrature rule. Then

$$\begin{aligned} \int_a^b f(x) dx - G_n(f) &= \int_a^b f(x) - p(x) dx + \\ &\quad \int_a^b p(x) dx - G_n(p) + G_n(p) - G_n(f). \end{aligned}$$

But because  $G_n(\cdot)$  is exact for polynomials of degree  $n$ ,  $\int_a^b p(x) dx - G_n(p) = 0$ . Using this, and the triangle inequality,

$$\begin{aligned} \left| \int_a^b f(x) dx - G_n(p) \right| &\leq \\ &\quad \left| \int_a^b f(x) - p(x) dx \right| + \left| G_n(p) - G_n(f) \right|. \end{aligned}$$

But

$$\left| \int_a^b f(x) - p(x) dx \right| \leq \int_a^b \epsilon dx = \epsilon(b - a).$$

Also,

$$\begin{aligned} |G_n(f) - G_n(p)| &= |G_n(p - f)| = \\ &= \left| \sum_{k=0}^n w_k (f(x_k) - p(x_k)) \right| = \\ &= \sum_{k=0}^n w_k |f(x_k) - p(x_k)|, \end{aligned}$$

because all the  $w_i$  are positive. Then, using that  $\sum_k w_k = b - a$  and  $|f(x) - p(x)| \leq \epsilon$ , we get

$$|G_n(f) - G_n(p)| \leq \epsilon(b - a).$$

Combining these we have shown that for any  $\epsilon$ , one can always find  $n$  such that

$$\left| \int_a^b f(x) dx - G_n(p) \right| \leq 2\epsilon(b - a).$$

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One of the key ingredients to this proof was that the  $w_i$  are all positive. That is not the case for Newton-Cotes methods—for large enough  $n$  their quadrature weights can be negative—so no similar result is possible.

### 3.6.3 Exercise

**Exercise 3.16.** Prove Theorem 3.6.1.

**Exercise 3.17.** ★ Show that it is impossible to choose  $n + 1$  quadrature points and weights so that the  $n + 1$ -point quadrature rule

$$\int_a^b f(x) dx \approx \sum_{k=0}^n w_k f(x_k)$$

has precision  $2n + 2$ .

*Hint: To show the method does not have precision  $2n + 2$ , you just need to give a an example of a single polynomial  $p$  of degree exactly  $2n + 2$  for which  $\int_a^b p(x) dx \neq \sum_{k=0}^n w_k f(x_k)$ .*