# Chapter 4

# Finite Element Methods for BVPs

## 4.1 Boundary value problems

In this final section of MA378 we consider numerical schemes for particular ordinary differential equations called *Boundary Value Problems (BVPs)*. The main differences between them and the initial value problems (IVPs) of MA385/530 are:

- The equation gives the solution to the differential equation *two* (boundary) points, not just one (initial) point.
- The second derivative of the solution *always* appears in the equation. The IVPs we studied usually only had first derivatives.
- We usually think of the independent variable, $x$, as representing *space* rather than time.

For some BVPs it is possible to write down the exact solution; an example of such a case is given below in Example 4.1.1. Usually, however, this is not easy, or, indeed, possible. So numerical methods are needed to given *approximate* solutions. Two of the most popular numerical methods are

- *Finite Difference Methods* (FDMs), where one approximates the differential equation, based on Taylor series expansions.
- *Finite Element Methods* (FEMs), where the solution space is approximated, using ideas related to interpolation.

Since FEMs are in keeping with the philosophy of this course, we will study them.

More details are in Chap. 14 of the textbook [SM03].

### 4.1.1 Boundary value problems

To formulate a model BVP we first define a differential operator

$$L(u) := -u''(x) + r(x)u(x). \tag{4.1}$$

Then the general form of a BVP is: *find a function $u$, defined on the interval $[a, b]$, such that*

$$\begin{aligned} L(u) = f(x) \quad &\text{for} \quad a < x < b, \\ u(a) = 0, u(b) = 0. \end{aligned} \tag{4.2}$$

We make the assumptions that $r$ and $f$ are continuous and have as many continuous derivatives as we would like. Furthermore we always have that $r(x) > 0$ for all $x \in [a, b]$.

Some BVPs are reasonably easy to solve by hand. That is, we can write down a solution in terms of elementary functions. That is certainly the case if the functions $r$ and $f$ are constant:

**Example 4.1.1.** Verify that, for any constants $c_0$ and $c_1$, $u(x) = c_0 e^{2x} + c_1 e^{-2x} + x/4$ is a solution to

$$-u''(x) + 4u(x) = x \quad \text{for} \quad 0 \leqslant x \leqslant 3, \tag{4.3}$$

<u>Take notes</u>:

If we have two boundary conditions, we can uniquely determine $c_0$ and $c_1$. See Exercise 4.2.

Unlike Example 4.1.1, most boundary value problems are difficult or impossible to solve analytically and so, in general, we need approximations. However, it can often be possible to obtain *qualitative* information about the solutions.

**Lemma 4.1.2** (Maximum Principle)**.** *Suppose that* $L$ *is as defined in* (4.1)*, with* $r(x) > 0$ *for all* $x$*, and* $u$ *is a function such that* $Lu \geqslant 0$ *on* $(a, b)$*. If* $u(a) \geqslant 0$*,* $u(b) \geqslant 0$*. Then* $u \geqslant 0$ *for all* $x \in [a, b]$*.*

**Proof:**

Take notes:

This lemma is as useful as it is simple. See the following lemma, and also Exercise 4.4

**Lemma 4.1.3.** *There is at most one solution to (4.2).*

Take notes:

### 4.1.2 Exercises

**Exercise 4.1.** Suppose, instead of the differential operator defined in (4.1), we had the more general one:

$$L_q(u) := -u''(x) + q(x)u'(x) + r(x)u(x).$$

Does this $L_q$ also satisfy a maximum principle? If so, provide a proof. If not, give a counter example.

**Exercise 4.2.** Verify that

$$u(x) = \frac{x}{4} + \frac{3e^6(e^{-2x} - e^{2x})}{4(e^{12} - 1)}$$

is the exact solution to (4.1.1) with the boundary conditions $u(0) = 0$, $u(3) = 0$,

**Exercise 4.3.** In this section of the course, we'll always assume homogeneous boundary conditions. That is, that $u(x) = 0$ at the boundaries. Suppose the problem we wish to solve is

$$-u''(x) + r(x)u(x) = f(x) \quad u(0) = \alpha, u(1) = \beta.$$

Show how to find a problem which has the same left-hand side as this one, homogeneous boundary conditions, and with a solution that differs from this one only by a known linear function.

**Exercise 4.4.** Suppose that $u$ solves

$$-u''(x) + r(x)u(x) = f(x) \quad \text{on } (0, 1),$$

and $u(0) = u(1) = 0$. Let $\rho$ be such $r(x) \geqslant \rho > 0$, and define

$$C = \max_{0 \leqslant x \leqslant 1} |f(x)|/\rho.$$

Prove that $u(x) \leqslant C$.