

## 4.2 The variational formulation

### 4.2.1 The idea

The key sequence of ideas for FEMs is

- (i) First replace the differential equation with an integral equation, using integration by parts to reduce the order of the derivatives. This is called the “*variational*” or “*weak*” form of the equation. (Details are below).
- (ii) If we had a candidate for the true solution to the differential equation, we could trial it by substituting it back into the BVP.
- (iii) But the set of possible solutions is infinitely large, so we can’t check them all.
- (iv) So we choose a much smaller subset, and look for the solution there. The space we will use is the space of *piecewise linear splines*, from Section 2.1.
- (v) For every value of the spline that we have to determine, we write down a version of the integral equation that must be satisfied.
- (vi) This will give us a linear system of equations to solve.
- (vii) A simple but clever idea shows that the approximation we find is the best possible one.

### 4.2.2 Where the solution lives

The boundary value problem (4.2) above equates the terms in  $u$  and  $u''$  with a function  $f$  that is continuous on  $(a, b)$ , so we must have that  $u$ ,  $u'$  and  $u''$  are all continuous on the interval  $(a, b)$ . That is,  $u$  belongs to the space of functions  $C^2(a, b)$ .

So we can state the problem (4.2) more precisely: *find*  $u \in C^2(a, b)$  *such that*

$$-u''(x) + r(x)u(x) = f(x) \quad \text{for all } x \text{ in } (a, b),$$

$$\text{and } u(a) = u(b) = 0.$$

There are (uncountably) many functions in the space  $C^2(a, b)$  (we say it is *infinite dimensional*) – far too many to check one-by-one. So we choose small subset of  $C^2(a, b)$  that has only finitely many members. The FEM will let us find the element of that subset that is “closest” to the true solution.

### 4.2.3 The variational/weak form

Before we give a numerical method for computing an approximate solution to a boundary value problem, we will rewrite it as an integral equation.

**Definition 4.2.1.** We define

- $C_0^2(a, b)$  to be the space of all functions in  $C^2(a, b)$  that are zero at  $x = a$  and  $x = b$ .
- the inner product:  $(u, v) := \int_a^b u(x)v(x)dx$ ; and the induced norm:  $\|u\|_2 := (u, u)^{1/2}$ ,
- and  $H_0^1(a, b)$  to be the space of (absolutely) continuous functions on  $[a, b]$  such that if  $w \in H_0^1(a, b)$  then  $w(a) = w(b) = 0$  and  $\|w'\|_2 < \infty$ . (We met similar space before in the section on cubic splines.)

Consider the Boundary Value Problem: *find*  $u \in C^2(a, b)$  *such that*

$$\begin{aligned} -u''(x) + r(x)u(x) &= f(x) \quad \text{on } (a, b), \\ u(a) &= u(b) = 0. \end{aligned} \tag{4.4}$$

Suppose that we have a solution  $u$  to this. Then,...

Take notes:

Restrict the set of possible  $v$  so that  $v(a) = v(b) = 0$  to get another way of stating (4.4): *Find*  $u$  *so that*

$$(u', v') + (ru, v) = (f, v)$$

$$\text{for all } v \text{ such that } v(a) = v(b) = 0.$$

However, in (4.4) we required that  $u$  be twice differentiable; that seems unnecessary here where we have the much *weaker* requirements that  $u'$  exist and be integrable.<sup>1</sup>

<sup>1</sup>Actually, a more correct interpretation of the expression *weak* formulation is that it does not necessarily involve classical “strong” derivatives, but a more general concept of the weak derivative based on Lebesgue integration. But this is beyond the scope of this course, and so we will ignore this technicality.

**Definition 4.2.2** (Variational formulation). The weak/-variational formulation of (4.4) is: Find  $u \in H_0^1(a, b)$  such that

$$A(u, v) = L(v) \quad \text{for all } v \in H_0^1(a, b). \quad (4.5)$$

where  $A(\cdot, \cdot)$  is the (symmetric) bilinear functional

$$A(u, v) := (u', v') + (ru, v),$$

and  $L(v)$  is the linear functional

$$L(v) = (f, v).$$

This  $A$  has several important properties. One of these is that, for any function  $w$ ,

$$A(w, w) \geq 0,$$

and

$$A(w, w) = 0 \iff w = 0.$$

In immediate consequence of this is the following lemma.

**Lemma 4.2.3.** *The variational problem (4.5) has at most one solution.*

Take notes:

#### 4.2.4 Exercise

**Exercise 4.5.** ★ Consider the differential equation:

$$-u''(x) = \exp(x+1), \text{ on } (0, 2), \text{ and } u(0) = u(2) = 0.$$

- (i) State the variational formulation of this differential equation.
- (ii) Show that the solution to the variational problem is unique.