4.4 Error analysis

4.4.1 Cea's Lemma

We now show that the member of S found by the FEM is the "closest" to the true solution.

Lemma 4.4.1 (Cea's Lemma). *Let u be the solution to (4.5) and let u_h be the solution to (4.6).*

(i) The difference between the true and approximate solutions is orthogonal to S, i.e.,

$$\mathcal{A}(u - u_h, v_h) = 0 \text{ for all } v_h \in S,$$

and

(ii) There is no element of S that is closer to u than u_h:

$$\mathcal{A}(u - u_h, u - u_h) = \min_{v_h \in S} \mathcal{A}(u - v_h, u - v_h),$$

Proof. (See also Theorem 14.6 of [SM03])
Take notes:

□

Since $\mathcal{A}(\cdot, \cdot)$ is an inner product (see Definition 3.5.2) it induces a *norm:*

$$|||u||| := \sqrt{\mathcal{A}(u, u)}.$$

So we can write (ii) of Cea's Lemma as

$$|||u - u_h||| \leqslant |||u - v_h||| \qquad \text{for all } v_h \in S.$$

4.4.2 An example

This is as far as we will take the analysis. With a bit more work (and a little Fourier analysis) we could show that

$$\|u - u_h\|_2 \leqslant Ch^2 \|u''\|_2.$$

That is, the error is proportional to h^2. We can then further deduce that the method converges:

$$\lim_{h \to 0} \|u - u_h\|_2 = 0.$$

In place of a rigorous analysis, let us reason as follows. Let l be the piecewise linear interpolant to u as described in Section 2.1). Note that l belongs to S. So, u_h as at least as good an approximation to u as l. That is

$$\|u - u_h\|_2 \leqslant \|u - l\|_2$$

And Theorem 2.1.3 told us that

$$\|u - l\|_\infty \leqslant \frac{h^2}{8} \|u''\|_\infty.$$

So, if you believe that

$$\|u - l\|_2 \approx \|u - l\|_\infty,$$

Then it will follow that

$$\|u - u_h\|_2 \lessapprox Ch^2,$$

for some constant C. One can also show that

$$|||u - u_h||| \lessapprox Ch.$$

However that we have used three different norms here. Therefore some more work would be required to prove a rigorous result.

In Table 4.1 we shown the maximum error, over all mesh points, in the finite element solution to (4.9). One can see that the error is proportional to n^{-2} (and thus to h^2). See also Figure 4.2

These results were generated by a MATLAB program, which you can download from www.maths.nuigalway.ie/ niall/MA378/fe.m

Table 4.1: $\|u - u_h\|_\infty$ when solving (4.9)

n	$\|u - u_h\|_\infty$
4	2.908e-02
8	6.446e-03
16	1.629e-03
32	4.043e-04
64	1.009e-04
128	2.522e-05
256	6.304e-06
512	1.576e-06
1024	3.940e-07

4.4.3 Exercises

Exercise 4.9. Suppose that we want to solve

$$-u''(x) + u'(x) = 1 \text{ on } (a, b),$$

(a) Write down the system of linear equations that we would have to solve in terms of h.

(b) Does the analysis of Theorem 4.4.1 still hold? That is, can we use a similar argument to show that

$$|||u - u_h||| \leqslant |||u - v_h||| \quad \text{for all } v_h \in S?$$

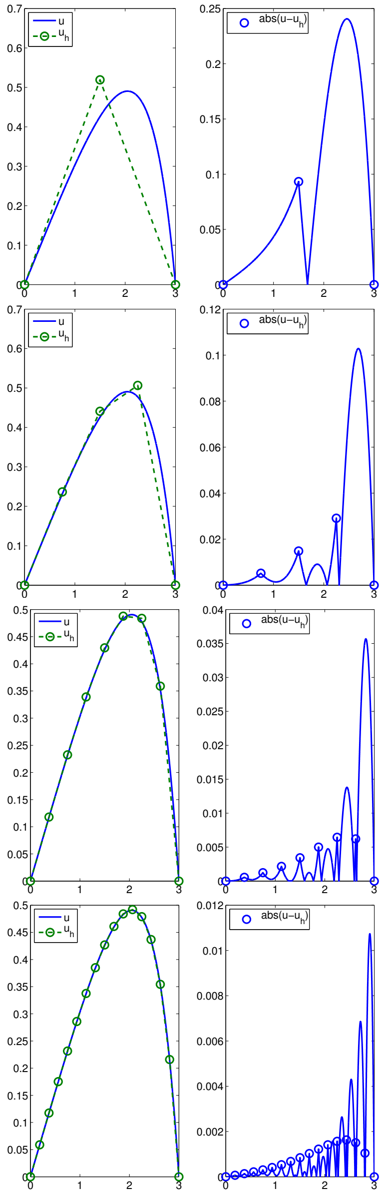

Fig. 4.2: Computed solutions and errors solving (4.9) with (from top) $n = 2, 4, 8, 16$

Exercise 4.10. Show that, for any function $f \in C^2[a, b]$,

$$\|f\|_2 \leqslant \sqrt{b-a}\,\|f\|_\infty,$$

where

$$\|f\|_2 := \left(\int_a^b (f(x))^2 dx\right)^{1/2} = \sqrt{(f, f)},$$

and

$$\|f\|_\infty := \max_{a \leqslant x \leqslant b} |f(x)|.$$

Exercise 4.10 shows that if we have a bound for $\|f\|_\infty$, we can get one for $\|f\|_2$. However, as the next exercise shows, the converse is not true.

Exercise 4.11. Show that, given any $\epsilon > 0$, no matter how small, it is possible to construct a function $f \in C^2[a, b]$, for which

$$\|f\|_2 \leqslant \varepsilon$$

but

$$\|f\|_\infty = 1.$$

4.5 FE Wrap-Up

There are many aspects of finite element methods that we did not cover, including

1. Finite element methods are *the most commonly used methods* for solving problems formulated as differential equations.

2. There are many other choices of basis functions given here. One could used cubic splines, or, indeed, higher-order polynomials.

3. When we try to improve the accuracy of the method by reducing h, this is called a h-FEM (and is the most common type).

4. We can also try to improve the accuracy of the method by increasing the order of the polynomials. This is called a p-FEM.

5. The ideas presented here extend to far more general problems. In particular, they work very well for problems in higher dimensions, and on weird-shaped domains.

The end!

I hope you enjoyed the course and now feel confident in your abilities as a Numerical Mathematician. Remember, if you even need to approximate a function, estimate an integral, derive a discrete derivative, or find a numerical solution to a boundary value differential equation, just reach for the nearest polynomial.

NM. March 2017