Chapter 4

Finite Element Methods for BVPs

4.1 Boundary value problems

In this final section of MA378 we consider numerical schemes for particular ordinary differential equations called *Boundary Value Problems (BVPs)*. The main differences between them and the initial value problems (IVPs) of MA385/530 are:

- The equation gives the solution to the differential equation *two* (boundary) points, not just one (initial) point.
- The second derivative of the solution *always* appears in the equation. The IVPs we studied usually only had first derivatives.
- We usually think of the independent variable, x, as representing *space* rather than time.

For some BVPs it is possible to write down the exact solution; an example of such a case is given below in Example 4.1.1. Usually, however, this is not easy, or, indeed, possible. So numerical methods are needed to given *approximate* solutions. Two of the most popular numerical methods are

- *Finite Difference Methods* (FDMs), where one approximates the differential equation, based on Taylor series expansions.
- *Finite Element Methods* (FEMs), where the solution space is approximated, using ideas related to interpolation.

Since FEMs are in keeping with the philosophy of this course, we will study them.

More details are in Chap. 14 of the textbook [SM03].

4.1.1 Boundary value problems

To formulate a model BVP we first define a differential operator

$$L(u) := -u''(x) + r(x)u(x). \tag{4.1}$$

Then the general form of a BVP is: *find a function* u*, defined on the interval* $[a, b]$*, such that*

$$\begin{aligned} L(u) = f(x) \quad &\text{for} \quad a < x < b, \\ u(a) = 0, u(b) &= 0. \end{aligned} \tag{4.2}$$

We make the assumptions that r and f are continuous and have as many continuous derivatives as we would like. Furthermore we always have that $r(x) > 0$ for all $x \in [a, b]$.

Some BVPs are reasonably easy to solve by hand. That is, we can write down a solution in terms of elementary functions. That is certainly the case if the functions r and f are constant:

Example 4.1.1. Verify that, for any constants c_0 and c_1, $u(x) = c_0e^{2x} + c_1e^{-2x} + x/4$ is a solution to

$$-u''(x) + 4u(x) = x \quad \text{for} \quad 0 \leqslant x \leqslant 3, \tag{4.3}$$

<u>Take notes</u>:

If we have two boundary conditions, we can uniquely determine c_0 and c_1. See Exercise 4.2.

Unlike Example 4.1.1, most boundary value problems are difficult or impossible to solve analytically and so, in general, we need approximations. However, it can often be possible to obtain *qualitative* information about the solutions.

Lemma 4.1.2 (Maximum Principle)**.** *Suppose that* L *is as defined in* (4.1)*, with* $r(x) > 0$ *for all* x*, and* u *is a function such that* $Lu \geqslant 0$ *on* (a, b)*. If* $u(a) \geqslant 0$*,* $u(b) \geqslant 0$*. Then* $u \geqslant 0$ *for all* $x \in [a, b]$*.*

Proof:

Take notes:

This lemma is as useful as it is simple. See the following lemma, and also Exercise 4.4

Lemma 4.1.3. *There is at most one solution to (4.2).*

Take notes:

4.1.2 Exercises

Exercise 4.1. Suppose, instead of the differential operator defined in (4.1), we had the more general one:

$$L_q(u) := -u''(x) + q(x)u'(x) + r(x)u(x).$$

Does this L_q also satisfy a maximum principle? If so, provide a proof. If not, give a counter example.

Exercise 4.2. Verify that

$$u(x) = \frac{x}{4} + \frac{3e^6(e^{-2x} - e^{2x})}{4(e^{12} - 1)}$$

is the exact solution to (4.1.1) with the boundary conditions $u(0) = 0$, $u(3) = 0$,

Exercise 4.3. In this section of the course, we'll always assume homogeneous boundary conditions. That is, that $u(x) = 0$ at the boundaries. Suppose the problem we wish to solve is

$$-u''(x) + r(x)u(x) = f(x) \quad u(0) = \alpha, u(1) = \beta.$$

Show how to find a problem which has the same left-hand side as this one, homogeneous boundary conditions, and with a solution that differs from this one only by a known linear function.

Exercise 4.4. Suppose that u solves

$$-u''(x) + r(x)u(x) = f(x) \quad \text{on } (0, 1),$$

and $u(0) = u(1) = 0$. Let ρ be such $r(x) \geqslant \rho > 0$, and define

$$C = \max_{0 \leqslant x \leqslant 1} |f(x)|/\rho.$$

Prove that $u(x) \leqslant C$.

4.2 The variational formulation

4.2.1 The idea

The key sequence of ideas for FEMs is

(i) First replace the differential equation with an integral equation, using integration by parts to reduce the order of the derivatives. This is called the "*variational*" or "*weak*" form of the equation. (Details are below).

(ii) If we had a candidate for the true solution to the differential equation, we could trial it by substituting it back into the BVP.

(iii) But the set of possible solutions is infinitely large, so we can't check them all.

(iv) So we choose a much smaller subset, and look for the solution there. The space we will use is the space of *piecewise linear splines*, from Section 2.1.

(v) For every value of the spline that we have to determine, we write down a version of the integral equation that must be satisfied.

(vi) This will give us a linear system of equations to solve.

(vii) A simple but clever idea shows that the approximation we find is the best possible one.

4.2.2 Where the solution lives

The boundary value problem (4.2) above equates the terms in u and u'' with a function f that is continuous on (a, b), so we must have that u, u' and u'' are all continuous on the interval (a, b). That is, u belongs to the space of functions $C^2(a, b)$.

So we can state the problem (4.2) more precisely: *find* $u \in C^2(a, b)$ *such that*

$$-u''(x) + r(x)u(x) = f(x) \quad \textit{for all } x \textit{ in } (a, b),$$

$$\textit{and } u(a) = u(b) = 0.$$

There are (uncountably) many functions in the space $C^2(a, b)$ (we say it is *infinite dimensional*) – far too many to check one-by-one. So we choose small subset of $C^2(a, b)$ that has only finitely many members. The FEM will let us find the element of that subset that is "closest" to the true solution.

4.2.3 The variational/weak form

Before we give a numerical method for computing an approximate solution to a boundary value problem, we will rewrite it as an integral equation.

Definition 4.2.1. We define

- $C_0^2(a, b)$ to be the space of all functions in $C^2(a, b)$ that are zero at $x = a$ and $x = b$.
- the inner product: $(u, v) := \int_a^b u(x)v(x)dx$; and the induced norm: $\|u\|_2 := (u, u)^{1/2}$,
- and $H_0^1(a, b)$ to be the space of (absolutely) continuous functions on $[a, b]$ such that if $w \in H_0^1(a, b)$ then $w(a) = w(b) = 0$ and $\|w'\|_2 < \infty$. (We met similar space before in the section on cubic splines.)

Consider the Boundary Value Problem: *find* $u \in C^2(a, b)$ *such that*

$$\begin{aligned} -u''(x) + r(x)u(x) = f(x) \quad &\textit{on } (a, b), \\ u(a) = u(b) = 0. & \end{aligned} \tag{4.4}$$

Suppose that we have a solution u to this. Then,...

`Take notes`:

Restrict the set of possible v so that $v(a) = v(b) = 0$ to get another way of stating (4.4): *Find* u *so that*

$$(u', v') + (ru, v) = (f, v)$$

$$\text{for all } v \text{ such that } v(a) = v(b) = 0.$$

However, in (4.4) we required that u be twice differentiable; that seems unnecessary here where we have the much *weaker* requirements that u' exist and be integrable.[1]

[1] Actually, a more correct interpretation of the expression *weak*" formulation is that it does not necessarily involve classical "strong" derivatives, but a more general concept of the weak derivative based on Lebesgue integration. But this is beyond the scope of this course, and so we will ignore this technicality.

Definition 4.2.2 (Variational formulation)**.** The weak/-variational formulation of (4.4) is: *Find* $u \in H_0^1(a, b)$ *such that*

$$A(u, v) = L(v) \quad \textit{for all} \quad v \in H_0^1(a, b). \tag{4.5}$$

where $\mathcal{A}(\cdot, \cdot)$ is the (symmetric) bilinear functional

$$\mathcal{A}(u, v) := (u', v') + (ru, v),$$

and $L(v)$ is the linear functional

$$L(v) = (f, v).$$

This $\mathcal{A}$ has several important properties. One of these is that, for any function w,

$$A(w, w) \geqslant 0,$$

and

$$A(w, w) = 0 \iff w = 0.$$

In immediate consequence of this is the following lemma.

Lemma 4.2.3. *The variational problem* (4.5) *has at most one solution.*

Take notes:

4.2.4 Exercise

Exercise 4.5. $\star$ Consider the differential equation:

$$-u''(x) = \exp(x+1), \text{ on } (0, 2), \text{ and } u(0) = u(2) = 0.$$

(i) State the variational formulation of this differential equation.

(ii) Show that the solution to the variational problem is unique.

4.3 The FEM

4.3.1 On Infinite- and Finite-Dimensional Spaces

A *space* is a collection of functions. The *dimension* of that space is smallest number of pieces of information that is required to uniquely describe any of its members.

Example 4.3.1. The set of points in $\mathbb{R}^2$. Each member can be written as (x_1, x_2) so we need two pieces of information ("coordinates") to describe the point. So $\mathbb{R}^2$ has dimension 2.

Example 4.3.2. The space of quadratic polynomials (over the reals). Each member can be written as $a+bx+cx^2$. So we need to know a, b and c for each member of the set; the space has dimension 3.

Another way of thinking about this is considering that these space have certain members that can be combined is various ways to give us every other member; they form a *basis* for the space.
Examples of bases for $\mathbb{R}^2$ and $\mathcal{P}^2$:
Take notes:

Example 4.3.3. All the solutions to the differential equation $u''(x) + u(x) = 0$ (note: no boundary conditions) can be written in the form $u(x) = \alpha \sin(x) + \beta \cos(x)$. So $\{\sin(x), \cos(x)\}$ is a basis for this space which is of dimension 2.

All of the spaces mentioned have *finite-dimension*. But consider the space of *all continuous functions on* $[a, b]$. This is an *infinite dimensional space*: we would have to write down an infinite number of values to describe a member uniquely.

Example 4.3.4. The space of functions $C^2(a, b)$ is infinite-dimensional, as is $C_0^2(a, b)$. So too is $H_0^1(a, b)$.

Example 4.3.5. The space of functions that are of the form $g(x) = \gamma(x-a)(x-b)$ for any $\gamma \in \mathbb{R}$ is *finite-dimensional* (because it has dimension 1). But every member of this space also belongs to the infinite dimensional space $H_0^1(a, b)$; it is a *finite-dimensional subspace of* $H_0^1(a, b)$.

4.3.2 The Galerkin Basis Functions

Example 4.3.6. To get another **very important** example of a finite dimensional subspace of $H_0^1(a, b)$, first fix a "*mesh*" on $[a, b]$. This is just a set of points $\{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$. Then consider the space of all functions that are *piecewise linear* on this mesh and that vanish at $x = a$ and $x = b$.

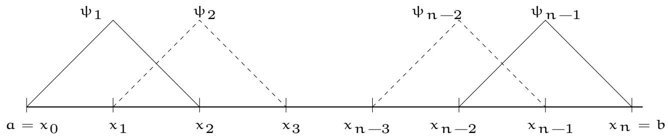
This is a a finite-dimensional sub-space of $H_0^1(a, b)$. A reasonable basis for this space would be the hat functions $\{\psi_1, \psi_2, \dots, \psi_{n-1}\}$ given by

$$\psi_i(x) = \begin{cases} (x - x_{i-1})/h & x_{i-1} \leqslant x < x_i \\ (x_{i+1} - x)/h & x_i \leqslant x \leqslant x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

where $h = (b-a)/n$ is the distance between adjacent points. Then we can write any function u_h as

$$u_h(x) = \lambda_1\psi_1(x) + \lambda_2\psi_2(x) + \cdots + \lambda_{n-1}\psi_{n-1}(x).$$

This basis set, shown below, are often called *hat functions* or *Galerkin*[2] *Basis functions*. We met them before in Section 2.1 on piecewise linear interpolation.



4.3.3 The Discrete Variational Form

Recall again that we are trying to solve the problem: *find* $u \in C^2(a, b)$

$$-u''(x) + r(x)u(x) = f(x) \quad \textit{on } (a, b),$$
$$u(a) = u(b) = 0,$$

where $r(x) > 0$ *for all* $x \in (a, b)$. We defined $\mathcal{A}(u, v) := (u', v') + (ru, v)$ and $L(v) := (f, v)$, and wrote the weak form of the DE as: *Find* $u \in H_0^1(a, b)$ *such that*

$$\mathcal{A}(u, v) = L(v) \quad \text{for all} \quad v \in H_0^1(a, b).$$

Now imagine we were trying to solve it by taking a function from $H_0^1(a, b)$ and checking if this integral equation is true for all functions v in $H_0^1(a, b)$. This would take forever because there are an infinite number of candidates. So instead we restrict our attention to a *finite-dimensional* subspace S of $H_0^1(a, b)$. Now we can select a function u_h from S and "put it on trial" by testing it against (all) the functions v_h in S. This leads to the terminology of calling u_h a *trial* function and v_h a *test* function.

Moreover, it leads to the following method.

[2]Born, 4 March 1871 in Polotsk, Belarus. Died 12 June 1945 in Moscow.

Definition 4.3.7 (The Finite Element Method)**.** Let S be the finite dimensional subspace of $H_0^1(a,b)$ made up of the piecewise linear functions on a fixed mesh $a = x_0 < x_1 < \cdots < x_n = b$. Then the *Galerkin Finite Element method* is: *find* $u_h \in S$ *such that*

$$\mathcal{A}(u_h, v_h) = (f, v_h) \qquad \textit{for all} \qquad v_h \in S. \tag{4.6}$$

4.3.4 FE implementation

We now want to look at how to turn Definition 4.3.7 into an algorithm (ideally, one that we can implement on a computer).

Let S be the space of piecewise linear functions on the mesh $x_i = a + ih$, where $h = (b-a)/n$. As above, u_h is can be written as

$$u_h(x) = \lambda_1\psi_1(x) + \lambda_2\psi_2(x) + \cdots + \lambda_{n-1}\psi_{n-1}(x).$$

So u_h has $n-1$ unknowns: $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$. To solve for these, we need $n-1$ equations. To get these, we just choose $n-1$ different (i.e., linearly independent) possible v_h, and substitute into (4.6).

The most obvious, and (it turns out) sensible, choice for these $n-1$ equations are the $n-1$ hat functions ψ_1, $\psi_2, \ldots, \psi_{n-1}$.

This gives us $n-1$ equations to solve:

$$\mathcal{A}(u_h, \psi_i) = (f, \psi_i) \qquad \text{for } i = 1, \ldots n-1. \tag{4.7}$$

It is now not too difficult to see that, if we write these equations as a matrix-vector equation, $Ax = F$, then

$$a_{i,j} = \mathcal{A}(\psi_i, \psi_j)$$

<u>`Take notes`</u>:

Some important observations:

- Any given "hat" function ψ_i is only non-zero on the region $[x_{i-1}, x_{i+1}]$.
- We have to compute

$$a_{i,j} = \mathcal{A}(\psi_i, \psi_j) = \int_a^b \psi_i'(x)\psi_j'(x) + r(x)\psi_i(x)\psi_j(x)dx. \tag{4.8}$$

But this will be non-zero only if there is overlap between $[x_{i-1}, x_{i+1}]$ and $[x_{j-1}, x_j{+}1]$.

- This gives that $A(\psi_i, \psi_j) = 0$ if $|i-j| > 1$, and otherwise

$$\mathcal{A}(\psi_i, \psi_j) = \int_{x_{i-1}}^{x_{i+1}} \psi_i'(x)\psi_j'(x) + r(x)\psi_i(x)\psi_j(x)dx.$$

Such a system is called *tridiagonal*. The left-hand side looks like this:

$$\begin{pmatrix} \mathcal{A}(\psi_1,\psi_1) & \mathcal{A}(\psi_2,\psi_1) & 0 & 0 & \cdots & 0 \\ \mathcal{A}(\psi_1,\psi_2) & \mathcal{A}(\psi_2,\psi_2) & \mathcal{A}(\psi_3,\psi_2) & 0 & \cdots & 0 \\ 0 & \mathcal{A}(\psi_3,\psi_2) & \mathcal{A}(\psi_3,\psi_3) & \mathcal{A}(\psi_3,\psi_4) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \end{pmatrix}$$

There are two consequences to this:

(i) It takes less effort to set up the linear systems of equations than one might have thought.

(ii) It is relatively easy to solve.

We should also observe the the matrix, A, is **symmetric**.

Note: in general a quadrature rule is used to compute in integrals

$$\int_{x_{i-1}}^{x_{i+1}} r(x)\psi_i(x)\psi_j(x).$$

The Gaussian Quadrature methods are the most popular for this.

An example

Use the FEM on the mesh $\{0, 1, 2, 3\}$ *to find an approximate solution to*

$$-u'' + 3u = x \quad \textit{on} \quad (0,3), \quad u(0) = u(3) = 0. \tag{4.9}$$

Solution: The FEM is: *Find* $u_h(x) \in S$ *such that*

$$\mathcal{A}(u_h, v_h) := (u_h', v_h') + 3(u_h, v_h) = (x, u_h) \qquad \textit{for all } v_h(x) \in S.$$

We have that $h = 1$ so let

$$\psi_1(x) = \begin{cases} x & 0 \leqslant x < 1 \\ 2-x & 1 \leqslant x \leqslant 2 \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_2(x) = \begin{cases} x-1 & 1 \leqslant x < 2 \\ 3-x & 2 \leqslant x \leqslant 3 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$u_h(x) = \lambda_1\psi_1(x) + \lambda_2\psi_2(x).$$

Our two equations are:

$$(\lambda_1\psi_1' + \lambda_2\psi_2', \psi_1') + 3(\lambda_1\psi_1 + \lambda_2\psi_2, \psi_1) = (x, \psi_1),$$
$$(\lambda_1\psi_1' + \lambda_2\psi_2', \psi_2') + 3(\lambda_1\psi_1 + \lambda_2\psi_2, \psi_2) = (x, \psi_2).$$

giving

$$\lambda_1\Big(\int_0^3 \psi_1'\psi_1'\,dx + 3\int_0^3 \psi_1\psi_1\,dx\Big) + \lambda_2\Big(\int_0^3 \psi_2'\psi_1'\,dx + 3\int_0^3 \psi_2\psi_1\,dx\Big) = \int_0^3 x\psi_1\,dx$$

$$\lambda_1\left(\int_0^3 \psi_1'\psi_2'\,dx + 3\int_0^3 \psi_1\psi_2\,dx\right) +$$
$$\lambda_2\left(\int_0^3 \psi_2'\psi_2'\,dx + 3\int_0^3 \psi_2\psi_2\,dx\right) = \int_0^3 x\psi_1\,dx.$$

We now need to evaluate these integrals. For example, from the 1st equation:

$$\int_0^3 \psi_1'\psi_1'\,dx = \int_0^1 (1)^2\,dx + \int_1^2 (-1)^2\,dx = 2,$$
$$\int_0^3 \psi_1\psi_1\,dx = \int_0^1 x^2\,dx + \int_1^2 (2-x)^2\,dx = 2/3,$$

so the coefficient of λ_1 is $2 + 3(2/3) = 4$. Also,

$$\int_0^3 \psi_2'\psi_1'\,dx = \int_1^2 \psi_2'\psi_1'\,dx = \int_1^2 (1)(-1)\,dx = -1,$$
$$\int_0^3 \psi_2\psi_1\,dx = \int_1^2 \psi_2\psi_1\,dx = \int_1^2 (x-1)(2-x)\,dx = 1/6.$$

So the coefficient of λ_2 is $-1 + 3(1/6) = -1/2$.

For the right-hand side:

$$\int_0^3 x\psi_1\,dx = \int_0^1 (x)(x)\,dx + \int_1^2 (x)(2-x)\,dx = 1/3 + 2/3 = 1.$$

Similarly,

$$\int_0^3 x\psi_2\,dx = 2.$$

The final system is

$$4\lambda_1 - \frac{1}{2}\lambda_2 = 1, \qquad -\frac{1}{2}\lambda_1 + 4\lambda_2 = 2.$$

This can also be written as a matrix-vector equation:

$$\begin{pmatrix} 4 & -\frac{1}{2} \\ -\frac{1}{2} & 4 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solving, we get $\lambda_1 = 20/63$ and $\lambda_2 = 34/63$. The solution is $u^h(x) = (20/63)\psi_1(x) + (34/63)\psi_2(x)$. This is shown in Figure 4.1

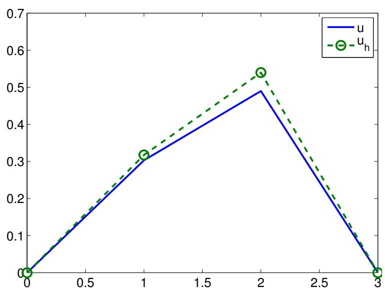


Fig. 4.1: The FE solution to (4.9) with $n = 3$

4.3.5 Exercises

Exercise 4.6. Show that u_h solves (4.7) *if and only if* it solves (4.6).

Exercise 4.7. Consider the problem:

$$-u''(x) = 9x \qquad u(0) = 0, u(1) = 0.$$

Use the FEM to find an approximate solution on the mesh $\{0, 1/3, 2/3, 1\}$.

Also write down the true solution to this problem.

Exercise 4.8. Suppose we want to use a finite element method to solve

$$-u''(x) + u(x) = 1 \text{ on } (0, 1),$$

with $u(0) = u(1) = 0$, using the usual piecewise linear basis functions on on uniform mesh $\{x_0, x_1, \ldots, x_n\}$. Let the resulting linear system is written as the matrix-vector equation $Au_h = F$.

(i) Show that the matrix A is symmetric (i.e. $a_{ij} = a_{ji}$).

(ii) Show that A is tridiagonal (i.e., if $|i - j| > 1$ then $a_{ij} = 0$).

(iii) Derive the formula for the entries of A in terms of h. That is, give an expression for $a_{i,i-1}$, $a_{i,i}$ and $a_{i,i+1}$.

4.4 Error analysis

4.4.1 Cea's Lemma

We now show that the member of S found by the FEM is the "closest" to the true solution.

Lemma 4.4.1 (Cea's Lemma)**.** *Let u be the solution to (4.5) and let u_h be the solution to (4.6).*

(i) The difference between the true and approximate solutions is orthogonal to S, i.e.,

$$\mathcal{A}(u - u_h, v_h) = 0 \text{ for all } v_h \in S,$$

and

(ii) There is no element of S that is closer to u than u_h:

$$\mathcal{A}(u - u_h, u - u_h) = \min_{v_h \in S} \mathcal{A}(u - v_h, u - v_h),$$

Proof. (See also Theorem 14.6 of [SM03])
Take notes:

□

Since $\mathcal{A}(\cdot, \cdot)$ is an inner product (see Definition 3.5.2) it induces a *norm:*

$$|||u||| := \sqrt{\mathcal{A}(u, u)}.$$

So we can write (ii) of Cea's Lemma as

$$|||u - u_h||| \leqslant |||u - v_h||| \qquad \text{for all } v_h \in S.$$

4.4.2 An example

This is as far as we will take the analysis. With a bit more work (and a little Fourier analysis) we could show that

$$\|u - u_h\|_2 \leqslant Ch^2 \|u''\|_2.$$

That is, the error is proportional to h^2. We can then further deduce that the method converges:

$$\lim_{h \to 0} \|u - u_h\|_2 = 0.$$

In place of a rigorous analysis, let us reason as follows. Let l be the piecewise linear interpolant to u as described in Section 2.1). Note that l belongs to S. So, u_h as at least as good an approximation to u as l. That is

$$\|u - u_h\|_2 \leqslant \|u - l\|_2$$

And Theorem 2.1.3 told us that

$$\|u - l\|_\infty \leqslant \frac{h^2}{8} \|u''\|_\infty.$$

So, if you believe that

$$\|u - l\|_2 \approx \|u - l\|_\infty,$$

Then it will follow that

$$\|u - u_h\|_2 \lessapprox Ch^2,$$

for some constant C. One can also show that

$$|||u - u_h||| \lessapprox Ch.$$

However that we have used three different norms here. Therefore some more work would be required to prove a rigorous result.

In Table 4.1 we shown the maximum error, over all mesh points, in the finite element solution to (4.9). One can see that the error is proportional to n^{-2} (and thus to h^2). See also Figure 4.2

These results were generated by a MATLAB program, which you can download from www.maths.nuigalway.ie/ niall/MA378/fe.m

Table 4.1: $\|u - u_h\|_\infty$ when solving (4.9)

n	$\|u - u_h\|_\infty$
4	2.908e-02
8	6.446e-03
16	1.629e-03
32	4.043e-04
64	1.009e-04
128	2.522e-05
256	6.304e-06
512	1.576e-06
1024	3.940e-07

4.4.3 Exercises

Exercise 4.9. Suppose that we want to solve

$$-u''(x) + u'(x) = 1 \text{ on } (a, b),$$

(a) Write down the system of linear equations that we would have to solve in terms of h.

(b) Does the analysis of Theorem 4.4.1 still hold? That is, can we use a similar argument to show that

$$|||u - u_h||| \leqslant |||u - v_h||| \quad \text{for all } v_h \in S?$$

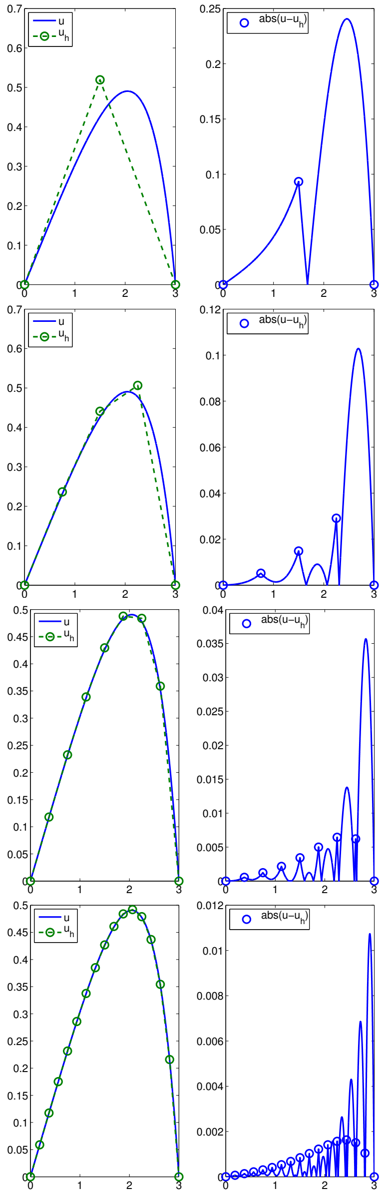


Fig. 4.2: Computed solutions and errors solving (4.9) with (from top) $n = 2, 4, 8, 16$

Exercise 4.10. Show that, for any function $f \in C^2[a, b]$,

$$\|f\|_2 \leqslant \sqrt{b-a}\|f\|_\infty,$$

where

$$\|f\|_2 := \left(\int_a^b (f(x))^2 dx\right)^{1/2} = \sqrt{(f, f)},$$

and

$$\|f\|_\infty := \max_{a \leqslant x \leqslant b} |f(x)|.$$

Exercise 4.10 shows that if we have a bound for $\|f\|_\infty$, we can get one for $\|f\|_2$. However, as the next exercise shows, the converse is not true.

Exercise 4.11. Show that, given any $\epsilon > 0$, no matter how small, it is possible to construct a function $f \in C^2[a, b]$, for which

$$\|f\|_2 \leqslant \varepsilon$$

but

$$\|f\|_\infty = 1.$$

4.5 FE Wrap-Up

There are many aspects of finite element methods that we did not cover, including

1. Finite element methods are *the most commonly used methods* for solving problems formulated as differential equations.

2. There are many other choices of basis functions given here. One could used cubic splines, or, indeed, higher-order polynomials.

3. When we try to improve the accuracy of the method by reducing h, this is called a h-FEM (and is the most common type).

4. We can also try to improve the accuracy of the method by increasing the order of the polynomials. This is called a p-FEM.

5. The ideas presented here extend to far more general problems. In particular, they work very well for problems in higher dimensions, and on weird-shaped domains.

The end!

I hope you enjoyed the course and now feel confident in your abilities as a Numerical Mathematician. Remember, if you even need to approximate a function, estimate an integral, derive a discrete derivative, or find a numerical solution to a boundary value differential equation, just reach for the nearest polynomial.

NM. March 2017