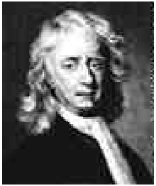
Solving nonlinear equations

(*See online notes and lecture notes for full details*)

§1.3: Newton's Method

MA385 – Numerical Analysis

September 2017



Sir Isaac Newton, 1643 - 1727, England. Easily one of the greatest scientist of all time. The method we are studying appeared in his celebrated *Principia Mathematica* in 1687, but it is believed he had used it as early as 1669.

Secant method can be written as

$$x_{k+1} = x_k - f(x_k)\phi(x_k, x_{k-1}),$$

where the function ϕ is chosen so that x_{k+1} is the root of the secant line joining the points $\big(x_{k-1}, f(x_{k-1})\big)$ and $\big(x_k, f(x_k)\big)$.

A related idea is to construct a method $x_{k+1} = x_k - f(x_k)\lambda(x_k)$, where we choose λ so that x_{k+1} is the point where the tangent line to f through $(x_k, f(x_k))$ cuts the x-axis.

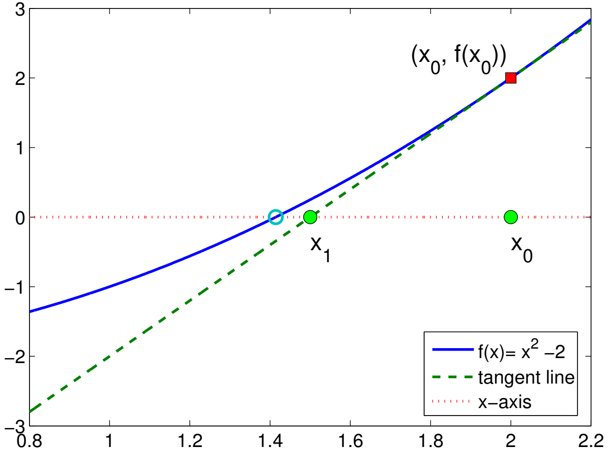


Figure: Estimating $\sqrt{2}$, by solving $x^2 - 2 = 0$ using Newton's Method

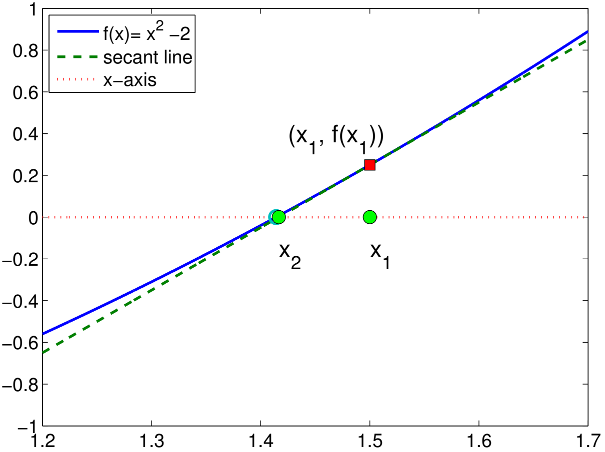


Figure: Estimating $\sqrt{2}$, by solving $x^2 - 2 = 0$ using Newton's Method

Method (Newton's Method)

1. Choose any x_0 in $[a, b]$,
2. For $i = 0, 1, \ldots,$ set x_{k+1} to the root of the line through x_k with slope $f'(x_k)$.

By writing down the equation for the line at $\left(x_k, f(x_k)\right)$ with slope $f'(x_k)$, we can show that the formula for the iteration is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{1}$$

Example

Use bisection, secant, and Newton's Method to solve $x^2 - 2 = 0$ *in* $[0, 2]$.

For this case, Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - 2}{2x_k},$$

which simplifies as

$$x_{k+1} = \frac{1}{2}x_k + \frac{1}{x_k}.$$

Taking $x_0 = 2$, we get $x_1 = 3/2$.
Then $x_2 = 2x_1 + 1/x_1 = 17/12 = 1.46667$.
Then $x_3 = 1.4142$, etc.

Iter	Bisection	Secant	Newton
k	$\lvert x_k - \tau\rvert$	$\lvert x_k - \tau\rvert$	$\lvert x_k - \tau\rvert$
0	1.41	1.41	5.86e-01
1	5.86e-01	5.86e-01	8.58e-02
2	4.14e-01	4.14e-01	2.45e-03
3	8.58e-02	8.09e-02	2.12e-06
4	1.64e-01	1.44e-02	1.59e-12
5	3.92e-02	4.20e-04	2.34e-16
6	2.33e-02	2.12e-06	—
7	7.96e-03	3.16e-10	—
8	7.66e-03	4.44e-16	—
9	1.51e-04	—	—
10	3.76e-03	—	—
11	1.80e-03	—	—
$\vdots$	$\vdots$	$\vdots$	$\vdots$
22	5.72e-07	—	—

Deriving Newton's method geometrically certainly has an intuitive appeal. However, to analyse the method, we need a more abstract derivation based on a **Truncated Taylor Series**.

$$
\begin{aligned}
f(x) = f(x_k) + (x - x_k)f'(x_k) + \frac{(x - x_k)^2}{2!}f''(x_k) + \dots \\
+ \frac{(x - x_k)^n}{n!}f^{(n)}(x_k) + \frac{(x - x_k)^{n+1}}{(n+1)!}f^{(n+1)}(\eta_k)
\end{aligned}
$$

where $\eta_k \in (x, x_k)$. Now truncate at the second term (i.e., take $n = 1$):

We now want to show that Newton's converges *quadratically*, that is, with *at least order* $q = 2$. To do this, we need to

1. Write down a recursive formula for the error.
2. Show that it converges.
3. Then find the limit of $\dfrac{|\tau - x_{k+1}|}{|\tau - x_k|^2}$.

Step 2 is usually the crucial part.

There are two parts to the proof. The first involves deriving the so-called "Newton Error formula".

We'll assume that the functions f, f' and f'' are defined and continuous on the an interval $I_\delta = [\tau - \delta, \tau + \delta]$ around the root τ.

The following proof is essentially the same as the above derivation (see also Theorem 3.2 in Epperson).

Theorem (Newton Error Formula)

If $f(\tau) = 0$ *and*

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},$$

then there is a point η_k *between* τ *and* x_k *such that*

$$\tau - x_{k+1} = -\frac{(\tau - x_k)^2}{2} \frac{f''(\eta_k)}{f'(x_k)},$$

Example

As an application of Newton's error formula, we'll show that the number of correct decimal digits in the approximation doubles at each step.

We'll now complete our analysis of this section by proving the convergence of Newton's method.

Theorem

Let us suppose that f is a function such that

- *f is continuous and real-valued, with continuous f'', defined on some close interval $I_\delta = [\tau - \delta, \tau + \delta]$,*
- *$f(\tau) = 0$ and $f''(\tau) \neq 0$,*
- *there is some positive constant A such that*

$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \text{for all} \quad x, y \in I_\delta.$$

Let $h = \min\{\delta, 1/A\}$. If $|\tau - x_0| \leq h$ then Newton's Method converges quadratically.