Solving nonlinear equations

(*See online notes and lecture notes for full details*)

§1.3: Newton's Method

MA385 – Numerical Analysis

September 2017

Sir Isaac Newton, 1643 - 1727, England. Easily one of the greatest scientist of all time. The method we are studying appeared in his celebrated *Principia Mathematica* in 1687, but it is believed he had used it as early as 1669.

Secant method can be written as

$$x_{k+1} = x_k - f(x_k)\phi(x_k, x_{k-1}),$$

where the function ϕ is chosen so that x_{k+1} is the root of the secant line joining the points $\big(x_{k-1}, f(x_{k-1})\big)$ and $\big(x_k, f(x_k)\big)$.

A related idea is to construct a method $x_{k+1} = x_k - f(x_k)\lambda(x_k)$, where we choose λ so that x_{k+1} is the point where the tangent line to f through $(x_k, f(x_k))$ cuts the x-axis.

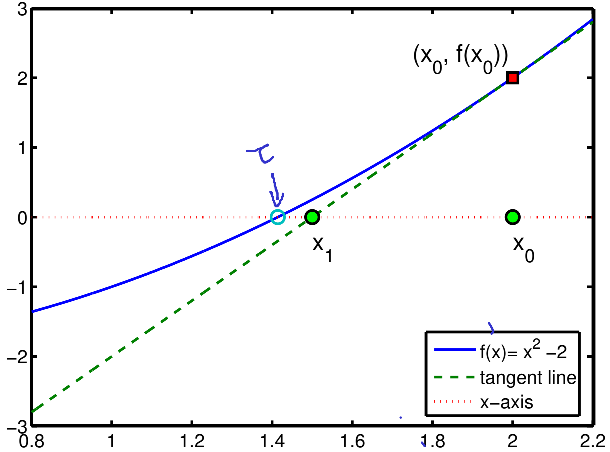

Figure: Estimating $\sqrt{2}$, by solving $x^2 - 2 = 0$ using Newton's Method

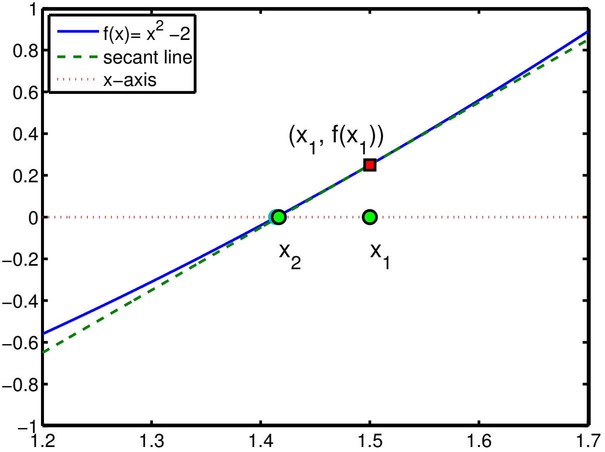

Figure: Estimating $\sqrt{2}$, by solving $x^2 - 2 = 0$ using Newton's Method

Method (Newton's Method)

1. Choose any x_0 in $[a, b]$,
2. For $i = 0, 1, \ldots,$ set x_{k+1} to the root of the line through x_k with slope $f'(x_k)$.

By writing down the equation for the line at $\big(x_k, f(x_k)\big)$ with slope $f'(x_k)$, we can show that the formula for the iteration is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{1}$$

Exer 1.7: Show this.

Example

Use bisection, secant, and Newton's Method to solve $\underbrace{x^2 - 2}_{f(x)} = 0$ *in* $[0, 2]$.

For this case, Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - 2}{2x_k},$$

which simplifies as

$$x_{k+1} = \frac{1}{2}x_k + \frac{1}{x_k}.$$

Taking $x_0 = 2$, we get $x_1 = 3/2$.
Then $x_2 = 2x_1 + 1/x_1 = 17/12 = 1.46667$.
Then $x_3 = 1.4142$, etc.

Iter	Bisection	Secant	Newton
k	$\lvert x_k - \tau \rvert$	$\lvert x_k - \tau \rvert$	$\lvert x_k - \tau \rvert$
0	1.41	1.41	5.86e-01
1	5.86e-01	5.86e-01	8.58e-02
2	4.14e-01	4.14e-01	2.45e-03
3	8.58e-02	8.09e-02	2.12e-06
4	1.64e-01	1.44e-02	1.59e-12
5	3.92e-02	4.20e-04	2.34e-16
6	2.33e-02	2.12e-06	—
7	7.96e-03	3.16e-10	—
8	7.66e-03	4.44e-16	—
9	1.51e-04	—	—
10	3.76e-03	—	—
11	1.80e-03	—	—
$\vdots$	$\vdots$	$\vdots$	$\vdots$
22	5.72e-07	—	—

Deriving Newton's method geometrically certainly has an intuitive appeal. However, to analyse the method, we need a more abstract derivation based on a **Truncated Taylor Series**.

Eta

$$f(x) = f(x_k) + (x - x_k)f'(x_k) + \frac{(x - x_k)^2}{2!}f''(x_k) + \dots + \frac{(x - x_k)^n}{n!}f^{(n)}(x_k) + \frac{(x - x_k)^{n+1}}{(n+1)!}f^{(n+1)}(\eta_k)$$

where $\eta_k \in (x, x_k)$. Now truncate at the second term (i.e., take $n = 1$): Take $\tau = x$, so

$$\overset{=0}{f(\tau)} = f(x_k) + (\tau - x_k) f'(x_k) + \tfrac{1}{2}(\tau - x_k)^2 f''(\eta_k)$$

So $\tau - x_k = -f(x_k)/f'(x_k) - \tfrac{1}{2}(\tau - x_k)^2 \dfrac{f''(\eta_k)}{f'(x_k)}$

Neglect the final term: $\tau \approxeq x_k - f(x_k)/f'(x_k)$

Newton Error Formula

We now want to show that Newton's converges *quadratically*, that is, with *at least order* $q = 2$. To do this, we need to

1. Write down a recursive formula for the error. → Newton Error Formula (NEF)
2. Show that it converges.
3. Then find the limit of $\dfrac{|\tau - x_{k+1}|}{|\tau - x_k|^2}$.

Step 2 is usually the crucial part.

There are two parts to the proof. The first involves deriving the so-called "Newton Error formula".

We'll assume that the functions f, f' and f'' are defined and continuous on the an interval $I_\delta = [\tau - \delta, \tau + \delta]$ around the root τ.

The following proof is essentially the same as the above derivation (see also Theorem 3.2 in Epperson).

Theorem (Newton Error Formula)

If $f(\tau) = 0$ and

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},$$

then there is a point η_k between τ and x_k such that

$$\tau - x_{k+1} = -\frac{(\tau - x_k)^2}{2}\frac{f''(\eta_k)}{f'(x_k)},$$

From the Taylor series:

$$\overset{=0}{f(\tau)} = f(x_k) + (\tau - x_k) f'(x_k) + \tfrac{1}{2}(\tau - x_k)^2 f''(\eta_k).$$

Rearrange to get

$$\tau \underbrace{- x_k + f(x_k)/f'(x_k)}_{-x_{k+1}} = -\tfrac{1}{2}(\tau - x_k)^2 f''(\eta_k).$$

So $$\tau - x_{k+1} = -\tfrac{1}{2}(\tau - x_k)^2 f''(\eta_k).$$

Example

As an application of Newton's error formula, we'll show that the number of correct decimal digits in the approximation doubles at each step.

Idea: If x_k is the approximation, then $\tau - x_k$ is the error, and this has

$d_k = \log_{10} |\tau - x_k|$ digits.

FROM NEF:

$$\log_{10} |\tau - x_{k+1}| = 2 \log_{10} |\tau - x_k| + \log_{10} \left| \frac{f''(\eta_k)}{f'(x_k)} \right|$$

$$\Rightarrow \quad d_{k+1} = 2 d_k + b_k.$$

We'll now complete our analysis of this section by proving the convergence of Newton's method.

Theorem

Let us suppose that f is a function such that

- *f is continuous and real-valued, with continuous f'', defined on some close interval $I_\delta = [\tau - \delta, \tau + \delta]$,*
- *$f(\tau) = 0$ and $f''(\tau) \neq 0$,*
- *there is some positive constant A such that*

$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \text{for all} \quad x, y \in I_\delta.$$

Let $h = \min\{\delta, 1/A\}$. If $|\tau - x_0| \leq h$ then Newton's Method converges quadratically.

From NEF:

$$|\tau - x_{k+1}| \leq \frac{1}{2}|\tau - x_k|^2 A.$$

Suppose $|\tau - x_k| \leq h$.

Then $|\tau - x_{k+1}| \leq \frac{1}{2}|\tau - x_k|(h)(A)$

But $h \leq 1/A$

So $|\tau - x_{k+1}| \leq \frac{1}{2}|\tau - x_k|$

So, $|\tau - x_{k+1}| \leq (1/2)^k |\tau - x_0|$.

So, as $k \to \infty$ $|\tau - x_k| \to 0$. Method converges

Finally, since $x_k \to \tau$, & $\eta_k \in [\tau, x_k]$ so $\eta_k \to \tau$.

Therefore, the Newton Error Formula,

$$|\tau - x_{k+1}| = \frac{1}{2} |\tau - x_k|^2 \frac{|f''(\eta_k)|}{|f'(x_k)|}$$

gives

$$\lim_{k \to \infty} \frac{|\tau - x_{k+1}|}{|\tau - x_k|^2} = \mu$$, where μ is

the constant $\frac{1}{2} \frac{|f''(\tau)|}{|f'(\tau)|}$.