Solving nonlinear equations

See online and lecture notes for full details

§1.4: Fixed Point Iteration

MA385 – Numerical Analysis 1

September 2017

Newton's method can be considered to be a special case of a very general approach called *Fixed Point Iteration* or *Simple Iteration*.

The basic idea is:

> *If we want to solve $f(x) = 0$ in $[a, b]$, find a function $g(x)$ such that, if τ is such that $f(\tau) = 0$, then $g(\tau) = \tau$. Choose x_0 and set $x_{k+1} = g(x_k)$ for $k = 0, 1, 2, \ldots$.*

Example

Suppose that $f(x) = e^x - 2x - 1$ and we are trying to find a solution to $f(x) = 0$ in $[1, 2]$. Then we can take $g(x) = \ln(2x + 1)$.

If we take $x_0 = 1$, then we get the following sequence:

k	x_k	$\lvert\tau - x_k\rvert$
0	1.0000	2.564e-1
1	1.0986	1.578e-1
2	1.1623	9.415e-2
3	1.2013	5.509e-2
4	1.2246	3.187e-2
5	1.2381	1.831e-2
$\vdots$	$\vdots$	$\vdots$
10	1.2558	6.310e-4

We have to be quite careful with this method: **not every choice is g is suitable**.

For example, suppose we want the solution to $f(x) = x^2 - 2 = 0$ in $[1, 2]$. We could choose $g(x) = x^2 + x - 2$. Then, if take $x_0 = 1$ we get the sequence:

We need to refine the method that ensure that it will converge.

Before we do that in a formal way, consider the following...

Example

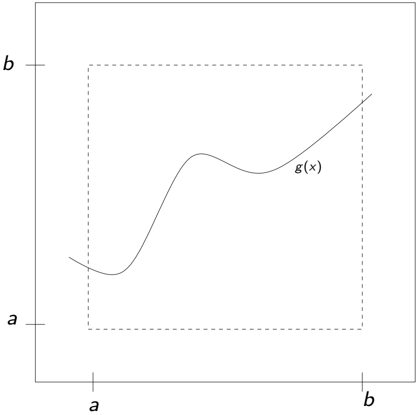
Use the Mean Value Theorem to show that the fixed point method $x_{k+1} = g(x_k)$ converges if $|g'(x)| < 1$ for all x near the fixed point.

This example:

- introduces the tricks of using that $g(\tau) = \tau$ & $g(x_k) = x_{k+1}$.
- Leads us towards the **contraction mapping theorem**.

Theorem (Fixed Point Theorem)

Suppose that $g(x)$ is defined and continuous on $[a, b]$, and that $g(x) \in [a, b]$ for all $x \in [a, b]$. Then there exists $\tau \in [a, b]$ such that $g(\tau) = \tau$. That is, $g(x)$ has a fixed point in $[a, b]$.



Next suppose that g is a *contraction*. That is, $g(x)$ is continuous and defined on $[a, b]$ and there is a number $L \in (0, 1)$ such that

$$|g(\alpha) - g(\beta)| \leq L|\alpha - \beta| \text{ for all } \alpha, \beta \in [a, b]. \tag{1}$$

Theorem (Contraction Mapping Theorem)

Suppose that the function g is a real-valued, defined, continuous, and

(a) *maps every point in $[a, b]$ to some point in $[a, b]$, and*

(b) *is a contraction on $[a, b]$,*

then

(i) $g(x)$ *has a fixed point $\tau \in [a, b]$,*

(ii) *the fixed point is unique,*

(iii) *the sequence $\{x_k\}_{k=0}^{\infty}$ defined by $x_0 \in [a, b]$ and $x_k = g(x_{k-1})$ for $k = 1, 2, \dots$ converges to τ.*

The algorithm generates as sequence $\{x_0, x_1, \ldots, x_k\}$. Eventually we must stop. Suppose we want the solution to be accurate to say 10^{-6}, how many steps are needed? That is, how big do we need to take k so that

$$|x_k - \tau| \leq 10^{-6}?$$

The answer is obtained by first showing that

$$|\tau - x_k| \leq \frac{L^k}{1-L}|x_1 - x_0|. \tag{2}$$

Example

Suppose we are using FPI to find the fixed point $\tau \in [1, 2]$ of $g(x) = \ln(2x + 1)$ with $x_0 = 1$, and we want $|x_k - \tau| \leq 10^{-6}$, then we can use (2) to determine the number of iterations required.