Solving nonlinear equations

See online and lecture notes for full details

§1.4: Fixed Point Iteration

MA385 – Numerical Analysis 1

September 2017

Newton's method can be considered to be a special case of a very general approach called *Fixed Point Iteration* or *Simple Iteration*.

The basic idea is:

> *If we want to solve* $f(x) = 0$ *in* $[a, b]$*, find a function* $g(x)$ *such that, if* τ *is such that* $f(\tau) = 0$*, then* $g(\tau) = \tau$*. Choose* x_0 *and set* $x_{k+1} = g(x_k)$ *for* $k = 0, 1, 2, \dots$ *.*

g has a fixed point where f has a zero.

Example

Suppose that $f(x) = e^x - 2x - 1$ and we are trying to find a solution to $f(x) = 0$ in $[1, 2]$. Then we can take $g(x) = \ln(2x + 1)$.

If we take $x_0 = 1$, then we get the following sequence:

k	x_k	$\|\tau - x_k\|$
0	1.0000	2.564e-1
1	1.0986	1.578e-1
2	1.1623	9.415e-2
3	1.2013	5.509e-2
4	1.2246	3.187e-2
5	1.2381	1.831e-2
⋮	⋮	⋮
10	1.2558	6.310e-4

The Error is strictly decreasing (but slowly).

We have to be quite careful with this method: **not every choice is g is suitable**.

For example, suppose we want the solution to $f(x) = x^2 - 2 = 0$ in $[1, 2]$. We could choose $g(x) = x^2 + x - 2$. Then, if take $x_0 = 1$ we get the sequence:

Then $x_{k+1} = g(x_k) = x_k^2 + x_k - 2$

k	x_k
0	$x_0 = 1$
1	$x_1 = 0^2 + 0 - 2 = -2$
2	$x_2 = (-2)^2 - 2 - 2 = 4 - 4 = 0$
3	$x_3 = 0^2 + 0 - 2 = -2$
4	0

This is not converging!

We need to refine the method that ensure that it will converge.

Before we do that in a formal way, consider the following...

Example

Use the Mean Value Theorem to show that the fixed point method $x_{k+1} = g(x_k)$ converges if $|g'(x)| < 1$ for all x near the fixed point.

Recall the MVT: $\frac{g(b) - g(a)}{b-a} = g'(c)$ for some $c \in [a,b]$.

So $|g(b) - g(a)| = |g'(c)| \cdot |b-a|$. Take $b = \tau$ & $a = x_k$, and then

$|g(\tau) - g(x_k)| = |g'(c)|\,|\tau - x_k|$. But $g(\tau) = \tau$ & $g(x_k) = x_{k+1}$

So $|\tau - x_{k+1}| = |g'(c)|\,|\tau - x_k|$. But $|g'(c)| < 1$, so

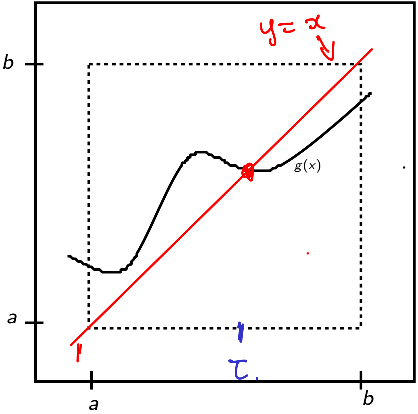
$$|\tau - x_{k+1}| < |\tau - x_k|.$$

This example:

- introduces the tricks of using that $g(\tau) = \tau$ & $g(x_k) = x_{k+1}$.
- Leads us towards the **contraction mapping theorem**.

Theorem (Fixed Point Theorem)

Suppose that $g(x)$ is defined and continuous on $[a, b]$, and that $g(x) \in [a, b]$ for all $x \in [a, b]$. Then there exists $\tau \in [a, b]$ such that $g(\tau) = \tau$. That is, $g(x)$ has a fixed point in $[a, b]$.



$a \le g(x) \le b$ for all x s.t. $a \le x \le b$

Proof: Let $h(x) = g(x) - x$.

So $h(a) = g(a) - a \geq 0$

and $h(b) = g(b) - b \le 0$.

So, by the intermediate value theorem, $\exists\, \tau \in [a, b]$ for which $h(\tau) = 0$. So $g(\tau) = \tau$.

i.e. $0 < L < 1$

Next suppose that g is a *contraction*. That is, $g(x)$ is continuous and defined on $[a, b]$ and there is a number $L \in (0, 1)$ such that

$$|g(\alpha) - g(\beta)| \leq L|\alpha - \beta| \text{ for all } \alpha, \beta \in [a, b]. \tag{1}$$

Theorem (Contraction Mapping Theorem)

Suppose that the function g is a real-valued, defined, continuous, and

(a) *maps every point in $[a, b]$ to some point in $[a, b]$, and*

(b) *is a contraction on $[a, b]$,*

then

(i) *$g(x)$ has a fixed point $\tau \in [a, b]$,*

This is the fixed point theorem we just proved.

(ii) *the fixed point is unique,*

(iii) *the sequence $\{x_k\}_{k=0}^{\infty}$ defined by $x_0 \in [a, b]$ and $x_k = g(x_{k-1})$ for $k = 1, 2, \ldots$ converges to τ.*

(ii) Suppose there are two fixed points, τ_1 & τ_2

So $|g(\tau_1) - g(\tau_2)| = |\tau_1 - \tau_2|$.

But since g is a contraction,

$$|g(\tau_1) - g(\tau_2)| \leq L\,|\tau_1 - \tau_2| \ \& \ 0 < L < 1.$$

So, $|\tau_1 - \tau_2| \leq L\,|\tau_1 - \tau_2|$.

$\Rightarrow |\tau_1 - \tau_2|(1 - L) \leq 0$.

But $0 < L < 1$, so $(1 - L) > 0$.

So $|\tau_1 - \tau_2| \leq 0$, ie $\tau_1 = \tau_2$.

(iii) See board. (then finished here Monday)

The algorithm generates as sequence $\{x_0, x_1, \ldots, x_k\}$. Eventually we must stop. Suppose we want the solution to be accurate to say 10^{-6}, how many steps are needed? That is, how big do we need to take k so that

$$|x_k - \tau| \leq 10^{-6}?$$

The answer is obtained by first showing that

$$|\tau - x_k| \leq \frac{L^k}{1-L}|x_1 - x_0|. \qquad (2)$$

First recall... $|x_k - \tau| \leq L^k |x_0 - \tau|$

But $|x_0 - \tau| = |x_0 - x_1 + x_1 - \tau| \leq |x_0 - x_1| + |x_1 - \tau|$

So $|x_0 - \tau| \leq |x_0 - x_1| + L|x_0 - \tau|$

So $|x_0 - \tau|(1-L) \leq |x_0 - x_1|$. So $|x_0 - \tau| \leq \frac{|x_0 - x_1|}{1-L}$

Thus $|x_k - \tau| \leq \left(L/(1-L)\right)|x_0 - x_1|$

Example

Suppose we are using FPI to find the fixed point $\tau \in [1, 2]$ of $g(x) = \ln(2x+1)$ with $x_0 = 1$, and we want $|x_k - \tau| \leq 10^{-6}$, then we can use (2) to determine the number of iterations required.

we want to iterate until $|\tau - x_k| \leq \frac{L}{1-L} |x_0 - x_1| \leq 10^{-6}$.

To find k, we need an estimate for L.

First note that $g'(x) \leq \frac{2}{2x+1}$.

By the MVT, $L \leq \max\limits_{1 \leq c \leq 2} |g'(c)| = 2/3$.

So

$$|x_k - \tau| \leq 3\left(\frac{2}{3}\right)^k \underbrace{|0.0986|}_{|x_0 - x_1|}$$

So take K such that $3\left(\frac{2}{3}\right)^k |0.0986| \leq 10^{-6}$.

gives $K \sim 31$