MA385: Class Test, 19 October 2017: outline solutions

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Q1. *For each of the three following functions, show that it satisfies a Lipschitz condition, with respect to y, on the corresponding domain, and give an upper-bound for L.* Recall that $f = f(t, y)$ is Liptschitz w.r.t y on a domain D if there exists $L \in (0, \infty)$ such that $|f(t, u) - f(t, v)| \leq L|u - v|$ for all (t, u) and $(t, v) \in D$.

(i) $f(t, y) = -y$ for $t > 0$. `Solution:` $|f(t, u) - f(t, v)| = | - u + v| \leq L|u - v|$ with $L = 1$. [10 Marks]

(ii) $f(t, y) = 3y/\sqrt{t}$ for $t \in [1, \infty)$. `Solution:` This question had a typo as posed in the test, so any attempt of it gets full marks. In any case... $|f(t, u) - f(t, v)| = |3vt^{-1/2} + 3vt^{-1/2}| = 3t^{-1/2}|u - v| \leq L|u - v|$ with $L = 3$, where here we've used that, since $0 < t < \infty$, so $t^{-1/2} \leq 1$. [10 Marks]

(iii) $f(t, y) = 2 + t\cos(y)$ for $1 \leq t \leq 2$. `Solution:` First note that $|f(t, u) - f(t, v)| \leq |2 + t\cos(u) - 2 - t\cos(v)| = t|\cos(u) - \cos(v)|$. Next, the Mean Value Theorem gives that there is a point $c \in [u, v]$ such that $-\sin(c) = (\cos(u) - \cos(v))/(u - v)$. Since $|\sin(x)| \leq 1$ for all x, it follows that $|\cos(u) - \cos(v)| \leq |u - v|$ for all u, v. To conclude then, $|f(t, u) - f(t, v)| \leq t|u - v| \leq 2|u - v|$ since $t \in [1, 2]$. So $L = 2$ [20 Marks]

Q2. *State Euler's method for computing an approximate solution to the initial value problem (IVP)*

$$y(t_0) = y_0, \qquad y'(t) = f(t, y) \text{ for } t > t_0. \tag{1}$$

`Solution:` Choose n equally spaced points $t_0, t_1, \ldots, t_n$ so that $h = t_{k+1} - t_k$. (Equivalently, choose h, and set $t_k = t_0 + kh$). Then set $y_0 = y(t_0)$, and $y_{k+1} = y_k + hf(t_k, y_k)$ for $k = 0, 1, \ldots, n$.

Show how it can be derived from a truncated Taylor series expansion. [20 Marks]

`Solution:` We'll write a Taylor series for $y(t_{k+1})$ about $y(t_k)$:

$$y(t_{k+1}) = y(t_k) + (t_{k+1} - t_k)y'(t_k) + \frac{1}{2}(t_{k+1} - t_k)f''(\eta),$$

for some $\eta \in [t_k, t_{k+1}]$. Now use that $t_{k+1} - t_k = h$ and $y'(t_k) = f\big(t_k, y(t_k)\big)$, to get that

$$y(t_{k+1}) = y(t_k) + hf\big(t_k, y(t_k)\big) + \frac{h}{2}f''(\eta).$$

Neglect the h^2 term, and use y_k to represent the resulting approximation of $y(t_k)$ to get the method.

Q3. *We learned in class that the global error of Euler's method satisfies*

$$|\mathcal{E}_n| \leq \frac{h}{2} \max_{t_0 \leq t \leq t_1} |y''(t)| \frac{1}{L}(e^{L(t_n - t_0)} - 1). \tag{2}$$

Consider the IVP:

$$y(0) = 2, \qquad y'(t) = -y(t).$$

Determine from the smallest value of n should we take to ensure the error at $t = 1$ is no more that 10^{-3}? (Note that you already calculated L in Q1–(i) above). [40 Marks]

`Solution:` We need to find a bound for $y''(t)$ on $[0, 1]$. To do this, we could either

- solve the ODE exactly, to find that $y(t) = 2e^{-t}$, which leads to $y''(t) = 2e^{-t}$ and, consequently, $|y''(t)| \leq 2$ for $0 \leq t \leq q$. Or
- from the ODE, $y''(t) = \frac{\partial}{\partial t}y'(t) = -\frac{\partial}{\partial t}y(t) = y(t)$. Also, from the ODE we know that $y(0) = 2$ and $y(t) < 0$ on $[0, 1]$, so $y(t) \leq 2$ on $[0, 1]$. So, again, we have that $|y''(t)| \leq 2$.

Now to ensure that $|\mathcal{E}_n| \leq 10^{-3}$, we need $\frac{h}{2} \max_{t_0 \leq t \leq t_1} |y''(t)| \frac{1}{L}(e^{L(t_n - t_0)} - 1) \leq 10^{-3}$. That is

$$\frac{h}{2}(2)\frac{1}{1}(e^{(1-0)} - 1) \leq 10^{-3}.$$

Rearranging, we need $h \leq (10^{-3})/(1.71828) = 5.8198 \times 10^{-4}$. Since $n = 1/h$ for this problem, $n \geq 1718.28$. So the answer is $\mathbf{n = 1719}$.