MA385 Sample class-test, October 2017: Outline solutions

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An *Initial Value Problem (IVP)* is a differential equation of the form:

$$y(t_0) = y_0, \qquad y'(t) = f(t, y) \text{ for } t > t_0, \tag{1}$$

1. (a) *State Euler's method for computing an approximate solution to an IVP.*
   Answer: Choose equally spaced points $t_0, t_1, \ldots, t_n$ so that $t_i - t_{i-1} = h = (t_n - t_0)/n$ for $i = 0, \ldots, n-1$. Let y_i denote the approximation for $y(t)$ at $t = t_i$. Set

   $$y_{i+1} = y_i + hf(t_i, y_i), \quad i = 0, 1, \ldots, n-1. \tag{2}$$

   (b) *Show how to motivate it with a Taylor series expansion, and derive an expression for the truncation error at step:*

   $$T_i := \frac{y(t_{i+1}) - y(t_i)}{h} - \Phi(t_i, y(t_i); h). \tag{3}$$

   Answer: from Taylor's Theorem, there exists a point $\eta \in (t_i, t_{i+1})$ such that

   $$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{1}{2}(t_{i+1} - t_i)^2 y''(\eta).$$

   Use that $y'(t_i) = f(t_i, y(t_i))$ and $t_{i+1} - t_i = h$, to get that

   $$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{1}{2}h^2 y''(\eta). \tag{4}$$

   Our first application of (4) is to derive the method. If we neglect that $\mathcal{O}(h^2)$ term, and denote the resulting approximate of $y(t_i)$ as y_i, we get

   $$y_{i+1} = y_i + hf(t_i, y_i).$$

   For the error, note that, for Euler's Method, we take $\Phi(t_i, y_i; h) = f(t_i, y_i)$ in (5). Then rearranging (4) gives

   $$\frac{y(t_{i+1}) - y(t_i)}{h} - f(t_i, y(t_i)) = \frac{1}{2}hy''(\eta) =: T_i. \tag{5}$$

2. *Suppose we want an approximation for the solution to the IVP*

   $$y(1) = 2, \qquad y'(t) = \frac{1}{1+y} \text{ for } t > 1. \tag{6}$$

   at the point $t = 2$.

   (a) *Show that* $y(t) \geq 2$ *for all* $t \geq 1$.
   Answer: we know that $y(1) = 2$. Also, since $y'(t) = 1/(1 + y) > 0$ for $t > 1$. So y is an increasing function. It follows that $y(t) \geq 2$ for all $t \geq 1$.

   (b) *Show that* (6) *has a solution. That is, show that* $f(t, y) = 1/(1 + y)$ *satisfies a* Lipschitz condition *with respect to* y *on for all* $y \geq 1$. *Give an estimate for the Lipschitz constant,* L.
   Answer: f is Lipschitz, if $|f(t, u) - f(t, v)| \leq L|u - v|$ for some L. Note that $\partial f/\partial y = -1/(1 + y^2)$. So, since $y(t) \geq 2$, we know that $|\partial f/\partial y| \leq 1/5$. Then, from the Mean Value Theorem,

   $$|f(t, u) - f(t, v)| \leq (1/5)|u - v|.$$

   So f is Lipschitz, can we can take $L = 1/5$.

   (c) *We know that the global error,* $\mathcal{E}_n = y(t_n) - t_n$, *satisfies*

   $$|\mathcal{E}_n| \leq \frac{T}{L}(e^{L(t_n - t_0)} - 1), \tag{7}$$

   where $T = \max_{i=0,1,\ldots n} |T_i|$. *Use this to give an upper bound for the error for Euler's method when* $n = 10$ *and* $t_n = 2$.
   Answer: combining (5) and (7), we see that

   $$|\mathcal{E}_n| \leq \frac{h}{2} \max_{t_0 \leq t \leq t_1} |y''(t)| \frac{1}{L}(e^{L(t_n - t_0)} - 1). \tag{8}$$

   We know L, h, t_0, and t_n in (7), but we need a bound for y''. For that, we can differentiate the DE:

   $$y''(t) = \frac{\partial f(t)}{\partial t} = -\frac{y'(t)}{(1 + y(t))^2} = -\frac{1}{(1 + y(t))^3}.$$

   So, again using that $y'(t) \geq 2$, we get $|y''(t)| \leq 1/9$ for all t. Using this in (8) gives $|\mathcal{E}_n| \leq 0.01366$.