Chap. 2: Initial Value Problems

See online and lecture notes for full details

§2.1: Introduction

MA385/530 – Numerical Analysis 1

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Annotated slides

(See Chap 6 of Epperson for the introduction, and Chapter 12 of Süli and Mayers for the rest).

The growth of some tumours can be modelled as

$$\frac{dR}{dt} = \quad R'(t) = -\frac{1}{3}S_i R(t) + \frac{2\lambda\sigma}{\mu R + \sqrt{\mu^2 R^2 + 4\sigma}},$$

subject to the initial condition $R(t_0) = a$, where R is the radius of the tumour at time t.

Clearly, it would be useful to know the value of R as certain times in the future. Though it's essentially impossible to solve for R exactly, we can accurately estimate it. In this section, we'll study techniques for this.

→ Algorithms to estimate $R(t)$

& Analysis to say how good that estimate is.

Initial Value Problems (IVPs)

Initial Value Problems (IVPs) are differential equations of the form: *Find $y(t)$ such that*

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = f(t,y) \text{ for } t > t_0, \qquad \text{and } y(t_0) = y_0. \tag{1}$$

Here $y' = f(t,y)$ is the *differential equation* and $y(t_0) = y_0$ is the *initial value*.

Some IVPs are easy to solve. For example:

$$y' = t^2 \quad \text{with } y(1) = 1.$$

Since $\frac{dy}{dt} = t^2$, so integrating we get $y = \frac{1}{3}t^3 + C$.

Since $y(1) = 1$ we can determine C by $\frac{1}{3}(1)^3 + C = 1$ so $C = \frac{2}{3}$.

Thus $y(t) = \frac{1}{3}t^3 + \frac{2}{3}$.

Most problems are much harder, and some don't have solutions at all. In many cases, it is possible to determine that a giving problem does indeed have a solution, even if we can't write it down. The idea is that the function f should be "Lipschitz", a notion closely related to that of a **contraction**.

Definition (Lipschitz Condition)

A function f satisfies a **Lipschitz Condition** (with respect to its second argument) in the rectangular region D if there is a positive real number L such that

$$|f(t,u) - f(t,v)| \leq L|u - v| \tag{2}$$

for all $(t,u) \in D$ and $(t,v) \in D$.

We say "f is Lipschitz", $\Rightarrow$ "$f(t,y)$ does not change too much w.r.t. y".

Example

For each of the following functions f, show that is satisfies a *Lipschitz condition*, and give an upper bound on the Lipschitz constant L.

(i) $f(t,y) = y/(1+t)^2$ for $0 \leq t \leq \infty$.

Note this would not be Lipschitz on, say, $[-2,\infty)$.

(ii) $f(t,y) = 4y - e^{-t}$ for all t.

(iii) $f(t,y) = -(1+t^2)y + \sin(t)$ for $1 \leq t \leq 2$.

(i) $$|f(t,u) - f(t,v)| = \left| \frac{u}{(1+t)^2} - \frac{v}{(1+t)^2} \right|$$

$$\leq \underbrace{\frac{1}{(1+t)^2}}_{L} |u - v|.$$

So f is Lipschitz, with $L = 1$, since $\frac{1}{(1+t)^2} \leq 1$ for all $t > 0$.

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(ii) $f(t, y) = 4y - e^{-t}$ for all t.
(iii) $f(t, y) = -(1+t^2)y + \sin(t)$ for $1 \le t \le 2$.

(ii) See Board.

(iii) $|f(t,u) - f(t,v)| = |-(1+t^2)u + \cancel{\sin(t)} + (1+t^2)v - \cancel{\sin(t)}|$

$= \underbrace{(1+t^2)}_{L.}|u-v|$. So f is Lipschitz with $L = 5$.

Proposition (Picard's)

Suppose that the real-valued function $f(t, y)$ is continuous for $t \in [t_0, t_M]$ and $y \in [y_0 - C, y_0 + C]$; that $|f(t, y_0)| \leq K$ for $t_0 \leq t \leq t_M$; and that f satisfies the Lipschitz condition (2). If

$$C \geq \frac{K}{L}\left(e^{L(t_M - t_0)} - 1\right),$$

then (1) us a unique solution on $[t_0, t_M]$. Furthermore

$$|y(t) - y(t_0)| \leq C \qquad t_0 \leq t \leq t_M.$$

You are not required to know this theorem for this course. However, it's important to be able to determine when a given f satisfies a Lipschitz condition.